

AN L_p -THEORY FOR A CLASS OF NON-LOCAL ELLIPTIC EQUATIONS RELATED TO NONSYMMETRIC MEASURABLE KERNELS

ILDOO KIM AND KYEONG-HUN KIM

ABSTRACT. We study the integro-differential operators L with kernels $K(y) = a(y)J(y)$, where $J(y)dy$ is a Lévy measure on \mathbb{R}^d (i.e. $\int_{\mathbb{R}^d} (1 \wedge |y|^2) J(y) dy < \infty$) and $a(y)$ is an only measurable function with positive lower and upper bounds. Under few additional conditions on $J(y)$, we prove the unique solvability of the equation $Lu - \lambda u = f$ in L_p -spaces and present some L_p -estimates of the solutions.

1. INTRODUCTION

There has been growing interest in the integro-differential equations related to pure jump processes owing to their applications in various models in physics, biology, economics, engineering and many others involving long-range jumps and interactions. In this article we study the non-local elliptic equations having the operators

$$Lu := \int_{\mathbb{R}^d} \left(u(x+y) - u(x) - y \cdot \nabla u(x) \chi(y) \right) K(x, y) dy,$$

and

$$\tilde{L}u := \int_{\mathbb{R}^d} \left(u(x+y) - u(x) - y \cdot \nabla u(x) 1_{|y|<1} \right) K(x, y) dy,$$

where the kernel $K(x, y) = a(y)J(y)$ depends only on y ,

$$\chi(y) = 0 \text{ if } \sigma \in (0, 1), \quad \chi(y) = 1_{|y|<1} \text{ if } \sigma = 1, \quad \chi(y) = 1 \text{ if } \sigma \in (1, 2].$$

The constant σ depends on $J(y)$ and is defined in (2.10). In particular, if $J(y) = c(d, \alpha)|y|^{-d-\alpha}$ for some $\alpha \in (0, 2)$ then $\sigma = \alpha$. Note that if $a(y)$ is symmetric then $\tilde{L} = L$, and in general we (formally) have

$$\tilde{L}u = Lu + b \cdot \nabla u,$$

where

$$b^i = - \int_{B_1} y^i a(y) J(y) dy \quad \text{if } \sigma \in (0, 1), \quad b^i = \int_{\mathbb{R}^d \setminus B_1} y^i a(y) J(y) dy \quad \text{if } \sigma \in (1, 2].$$

The main goal of this article is to prove the unique solvability of the equations

$$Lu - \lambda u = f \quad \text{and} \quad \tilde{L}u - \lambda u = f, \quad \lambda > 0 \tag{1.1}$$

2010 *Mathematics Subject Classification.* 35R09, 47G20.

Key words and phrases. Non-local elliptic equations, Integro-differential equations, Lévy processes, non-symmetric measurable kernels.

The research of the second author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (2013020522).

in appropriate L_p -spaces and present some L_p -estimates of the solutions. Here $p > 1$. If $p = 2$, the only condition we are assuming is that $a(y)$ has positive lower and upper bounds and $J(y)$ is rotationally invariant. If $p \neq 2$, we assume some additional conditions on $J(y)$, which are described in (1.5) and (1.6) below (also see Assumption 2.18).

Below is a short description on related L_p -theories. For other results such as the Harnack inequality and Hölder estimates we refer the readers to [4], [5], [8], [10] and [14]. If $K(x, y) = c(d, \alpha)|y|^{-d-\alpha}$, where $\alpha \in (0, 2)$ and $c(d, \alpha)$ is some normalization constant, then L becomes the fractional Laplacian operator $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$. For the fractional Laplacian operator, L_p -estimates can be easily obtained by the Fourier multiplier theory (for instance, [16]). In [2] L_p -estimates were obtained for elliptic equations with “symmetric” kernels, and an L_p -theory for the equation $Lu - \lambda u = f$ with measurable nonsymmetric kernel $K(x, y) = a(y)|y|^{-d-\alpha}$ was recently introduced in [9]. For parabolic equations, the authors of [12] handled the equations with the kernel $K(x, y) = a(x, y)|y|^{-d-\alpha}$ under the condition that the coefficient $a(x, y)$ is homogeneous of order zero in y and sufficiently smooth in y , but it is allowed that a also depends on x . Lately in [17], an L_p -regularity theory for parabolic equations was constructed for $J(y)$ satisfying

$$\nu_1^\alpha(B) \leq \int I_B(y)J(y)dy \leq \nu_2^\alpha(B) \quad \forall B \in \mathcal{B}(\mathbb{R}^d),$$

where $\nu_i^{(\alpha)}$ are Lévy measures taking the form

$$\nu_i^{(\alpha)}(B) := \int_{\mathbb{S}^{d-1}} \left(\int_0^\infty \frac{1_B(r\theta)dr}{r^{1+\alpha}} \right) S_i(d\theta), \quad (1.2)$$

with finite surface measures dS_i on \mathbb{S}^{d-1} . Since the same constant α is used for both $\nu_1^{(\alpha)}$ and $\nu_2^{(\alpha)}$, even the Lévy measure $J(y)$ related to the operator $\Delta^{\alpha_1/2} + \Delta^{\alpha_2/2}$ is not of type (1.2) if $\alpha_1 \neq \alpha_2$.

From the probabilistic point of view, the fractional Laplacian operator can be described as the infinitesimal generator of α -stable processes. That is,

$$\Delta^{\alpha/2}f(x) = \lim_{t \rightarrow 0^+} \frac{1}{t} \mathbb{E}[f(x + X_t) - f(x)], \quad f \in C_0^\infty$$

where X_t is an \mathbb{R}^d -valued Lévy process in a probability space (Ω, P) with the characteristic function $\mathbb{E}e^{i\lambda \cdot X_t} := \int_\Omega e^{i\lambda \cdot X_t} dP = e^{-t|\lambda|^\alpha}$. More generally, for any Bernstein function ϕ with $\phi(0+) = 0$ (equivalently, $\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t})\mu(dt)$ for some measure μ satisfying $\int_0^\infty (1 \wedge t)\mu(dt) < \infty$), the operator $\phi(\Delta)$ is the infinitesimal generator of the process $X_t := W_{S_t}$, where S_t is a subordinator (i.e. an increasing Lévy process satisfying $S_0 = 0$) with Laplace exponent ϕ (i.e. $\mathbb{E}e^{\lambda S_t} = \exp\{t\phi(\lambda)\}$) and W_t is a d -dimensional Brownian motion independent of S_t . Such process is called the subordinate Brownian motion. Actually ϕ is a Bernstein function with $\phi(0+) = 0$ if and only if it is a Laplace exponent of a subordinator. Furthermore, the relation

$$\phi(\Delta)f := -\phi(-\Delta)f = \int_{\mathbb{R}^d} (f(x+y) - f(x) - \nabla f(x) \cdot y\chi(y)) J(y) dy \quad (1.3)$$

holds with $j(|y|) := J(y)$ given by

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(dt). \quad (1.4)$$

For the equations with the kernel $K(x, y) = a(y)J(y)$, an L_p -estimate is obtained in aforementioned article [2] if $a(y)$ is symmetric. However to the best of our knowledge, if the coefficient $a(y)$ is only measurable and $J(y) \neq |y|^{-d-\alpha}$ then the L_p -estimate has not been known yet. In this article we extend [9] to the class of Lévy measures $J(y)$ satisfying the following two conditions: (i) there exists a constant α_0 , where $\alpha_0 \in (0, 1]$ if $\sigma \leq 1$ and $\alpha_0 \in (1, 2)$ if $\sigma > 1$, so that

$$\frac{j(t)}{j(s)} \leq N\left(\frac{s}{t}\right)^{d+\alpha_0}, \quad \forall 0 < s \leq t, \quad (1.5)$$

and, (ii) for any $t > 0$

$$1_{\sigma < 1} \int_{|y| \leq 1} |y|j(t|y|) dy + 1_{\sigma \geq 1} \int_{|y| \leq 1} |y|^2 j(t|y|) dz \leq Nj(t). \quad (1.6)$$

See Section 2 for few remarks on these conditions. It is easy to check that (1.5) and (1.6) are satisfied if there exists $\alpha \geq \alpha_0$ so that

$$(s/t)^{d+\alpha} j(s) \leq N_1 j(t) \leq N_2 (s/t)^{d+\alpha_0} j(s), \quad \forall 0 < s \leq t. \quad (1.7)$$

One can construct many interesting jump functions $j(t)$ satisfying (1.7). For example, (1.7) holds if $J(y)$ is defined from (1.3) and ϕ is one of the following (see Example 2.12 for details):

- (1) $\phi(\lambda) = \sum_{i=1}^n \lambda^{\alpha_i}$, $0 < \alpha_i < 1$;
- (2) $\phi(\lambda) = (\lambda + \lambda^\alpha)^\beta$, $\alpha, \beta \in (0, 1)$;
- (3) $\phi(\lambda) = \lambda^\alpha (\log(1 + \lambda))^\beta$, $\alpha \in (0, 1)$, $\beta \in (0, 1 - \alpha)$;
- (4) $\phi(\lambda) = \lambda^\alpha (\log(1 + \lambda))^{-\beta}$, $\alpha \in (0, 1)$, $\beta \in (0, \alpha)$;
- (5) $\phi(\lambda) = (\log(\cosh(\sqrt{\lambda})))^\alpha$, $\alpha \in (0, 1)$;
- (6) $\phi(\lambda) = (\log(\sinh(\sqrt{\lambda})) - \log \sqrt{\lambda})^\alpha$, $\alpha \in (0, 1)$.

In these cases, the jump function $j(r)$ is comparable to $r^{-d}\phi(r^{-2})$.

Our approach is borrowed from [9]. We estimate the sharp functions of the solutions and apply the Hardy-Littlewood theorem and the Fefferman-Stein theorem. This approach is typically used to treat the second-order PDEs with small BMO or VMO coefficients (for instance, see [11]). In [9] this method is applied to a non-local operator with the kernel $K(x, y) = a(y)|y|^{-d-\alpha}$. As in [9], our sharp function estimates are based on some Hölder estimates of solutions. The original idea of obtaining Hölder estimates is from [3]. Nonetheless, since we are considering much general $J(y)$ rather than $c(d, \alpha)|y|^{-d-\alpha}$, many new difficulties arise. In particular, our operators do not have the nice scaling property which is used in [11] and [9], and this cause many difficulties in the estimates.

The article is organized as follows. In Section 2 we introduce the main results. Section 3 contains the unique solvability in the L_2 -space. In Section 4 we establish some Hölder estimates of solutions. Using these estimates we obtain the sharp function and maximal function estimates in Section 5. In Section 6, the proofs of main results are given.

We finish the introduction with some notation. As usual \mathbb{R}^d stands for the Euclidean space of points $x = (x^1, \dots, x^d)$, $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$ and $B_r := B_r(0)$. For $i = 1, \dots, d$, multi-indices $\beta = (\beta_1, \dots, \beta_d)$, $\beta_i \in \{0, 1, 2, \dots\}$, and functions $u(x)$ we set

$$u_{x^i} = \frac{\partial u}{\partial x^i} = D_i u, \quad D^\beta u = D_1^{\beta_1} \cdots D_d^{\beta_d} u, \quad |\beta| = \beta_1 + \dots + \beta_d.$$

For an open set $U \subset \mathbb{R}^d$ and a nonnegative non-integer constant γ , by $C^\gamma(U)$ we denote the usual Hölder space. For a nonnegative integer n , we write $u \in C^n(U)$ if u is n -times continuously differentiable in U . By $C_0^n(U)$ (resp. $C_0^\infty(U)$) we denote the set of all functions in $C^n(U)$ (resp. $C^\infty(U)$) with compact supports. Similarly by $C_b^n(U)$ (resp. $C_b^\infty(U)$) we denote the set of functions in $C^n(U)$ (resp. $C^\infty(U)$) with bounded derivatives. The standard L_p -space on U with Lebesgue measure is denoted by $L_p(U)$. We simply use $L_p, C^n, C_b^n, C_0^n, C_b^\infty$, and C_0^∞ when $U = \mathbb{R}^d$. We use “:=” to denote a definition. $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. If we write $N = N(a, \dots, z)$, this means that the constant N depends only on a, \dots, z . The constant N may change from location to location, even within a line. By \mathcal{F} and \mathcal{F}^{-1} we denote the Fourier transform and the inverse Fourier transform, respectively. That is, $\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx$ and $\mathcal{F}^{-1}(f)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} f(\xi) d\xi$. For a Borel set $A \subset \mathbb{R}^d$, we use $|A|$ to denote its Lebesgue measure and by $I_A(x)$ we denote the indicator of A .

2. SETTING AND MAIN RESULTS

Throughout this article, we assume that $J(y)$ is rotationally invariant,

$$\nu \leq a(y) \leq \Lambda \quad (2.8)$$

for some constants $\nu, \Lambda > 0$, and

$$\int_{\mathbb{R}^d} (1 \wedge |y|^2) J(y) dy < \infty. \quad (2.9)$$

Let e_1 be a unit vector. Obviously, the condition that $J(y)$ is rotationally invariant can be replaced by the condition that $J(y)$ is comparable to $j(|y|) := J(|y|e_1)$, because $J(y)a(y) = j(|y|) \cdot a(y)J(y)j^{-1}(|y|) := j(|y|)\tilde{a}(y)$ and \tilde{a} also has positive lower and upper bounds.

Denote

$$\sigma := \inf\{\delta > 0 : \int_{|y| \leq 1} |y|^\delta J(y) dy < \infty\}, \quad (2.10)$$

$$\chi(y) = 0 \text{ if } \sigma \in (0, 1), \quad \chi(y) = 1_{B_1} \text{ if } \sigma = 1, \quad \chi(y) = 1 \text{ if } \sigma \in (1, 2].$$

Note that if $J(y) = c(d, \alpha)|y|^{-d-\alpha}$ for some $\alpha \in (0, 2)$ then we have $\sigma = \alpha$.

For $u \in C_b^2$ we introduce the non-local elliptic operators

$$\mathcal{A}u = \int_{\mathbb{R}^d} (u(x+y) - u(x) - y \cdot \nabla u(x)\chi(y)) J(y) dy,$$

$$Lu = \int_{\mathbb{R}^d} (u(x+y) - u(x) - y \cdot \nabla u(x)\chi(y)) a(y)J(y) dy,$$

$$\tilde{L}u = \int_{\mathbb{R}^d} (u(x+y) - u(x) - y \cdot \nabla u(x)I_{B_1}(y)) a(y)J(y) dy,$$

$$L^*u = \int_{\mathbb{R}^d} (u(x+y) - u(x) - y \cdot \nabla u(x)\chi(y)) a(-y)J(-y) dy,$$

and

$$\tilde{L}^*u = \int_{\mathbb{R}^d} (u(x+y) - u(x) - y \cdot \nabla u(x)I_{B_1}(y)) a(-y)J(-y) dy.$$

We start with a simple but interesting result, which will be used later in the proof of Theorem 2.21.

Lemma 2.1. *For any $p > 1$ and $\lambda > 0$,*

$$\|u\|_{L_p} \leq \frac{1}{\lambda} \|\tilde{L}u - \lambda u\|_{L_p}, \quad \forall u \in C_0^\infty.$$

Proof. Put

$$\Phi(\xi) := - \int_{\mathbb{R}^d} (e^{i\xi \cdot y} - 1 - i(y \cdot \xi)I_{B_1})a(-y)J(-y) dy$$

and

$$f := \tilde{L}u - \lambda u.$$

Since $a(-y)J(-y)$ is a Lévy measure (i.e. $\int_{\mathbb{R}^d} (1 \wedge |y|^2)a(-y)J(-y) dy < \infty$), there exists a Lévy process whose characteristic exponent is $-t\Phi(\xi)$ (for instance, see Corollary 1.4.6 of [1]). Denoting by $p_\Phi(t, dx)$ its law at t , we have

$$\int_{\mathbb{R}^d} e^{-i\xi \cdot x} p_\Phi(t, dx) = \int_{\mathbb{R}^d} e^{i(-\xi) \cdot x} p_\Phi(t, dx) = e^{-t\Phi(-\xi)}. \quad (2.11)$$

In non-probabilistic terminology it can be rephrased that if $\int_{\mathbb{R}^d} (1 \wedge |y|^2)a(-y)J(-y) dy < \infty$ then there exists a continuous measure-valued function $p_\Phi(t, dx)$ such that $p_\Phi(t, \mathbb{R}^d) = 1$ and (2.11) holds. Since

$$(-\Phi(-\xi) - \lambda)\mathcal{F}u = \mathcal{F}f$$

and $\operatorname{Re} \Phi(-\xi) \geq 0$, we have

$$\begin{aligned} \mathcal{F}u(\xi) &= - \frac{1}{\Phi(-\xi) + \lambda} \mathcal{F}f(\xi) \\ &= - \left(\int_0^\infty e^{-t\Phi(-\xi) - \lambda t} dt \mathcal{F}f(\xi) \right) \\ &= - \left(\int_0^\infty \int_{\mathbb{R}^d} e^{-i\xi \cdot x} p_\Phi(t, dx) e^{-\lambda t} dt \mathcal{F}f(\xi) \right) \\ &= - \mathcal{F} \left(\int_0^\infty (p_\Phi(t, \cdot) * f(x)) e^{-\lambda t} dt \right) (\xi). \end{aligned}$$

Therefore,

$$u(x) = - \int_0^\infty (p_\Phi(t, \cdot) * f) e^{-\lambda t} dt$$

and by Young's inequality,

$$\|u\|_{L_p} \leq \int_0^\infty \int_{\mathbb{R}^d} p_\Phi(t, dx) e^{-\lambda t} dt \|f\|_{L_p} \leq \frac{1}{\lambda} \|f\|_{L_p}.$$

Hence the lemma is proved. \square

Definition 2.2. We write $u \in \mathcal{H}_p^A$ if and only if there exists a sequence of functions $u_n \in C_0^\infty$ such that $u_n \rightarrow u$ in L_p and $\{\mathcal{A}u_n : n = 1, 2, \dots\}$ is a Cauchy sequence in L_p . By $\mathcal{A}u$ we denote the limit of $\mathcal{A}u_n$ in L_p .

Lemma 2.3. \mathcal{H}_p^A is a Banach space equipped with the norm

$$\|u\|_{\mathcal{H}_p^A} := \|u\|_{L_p} + \|\mathcal{A}u\|_{L_p}.$$

Proof. It is obvious. \square

Definition 2.4. We say that $u \in \mathcal{H}_p^A$ is a solution of the equation

$$Lu - \lambda u = f \quad \text{in } \mathbb{R}^d \quad (2.12)$$

if and only if there exists a sequence $\{u_n \in C_0^\infty\}$ such that u_n converges to u in \mathcal{H}_p^A and $Lu_n - \lambda u_n$ converges to f in L_p . Similarly, we consider the equation

$$\tilde{L}u - \lambda u = f \quad \text{in } \mathbb{R}^d \quad (2.13)$$

in the same sense.

Lemma 2.5 (Maximum principle). *Let $\lambda > 0$, $b(x)$ be an \mathbb{R}^d -valued bounded function on \mathbb{R}^d and u be a function in C_b^2 satisfying $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$. If $Lu + b(x) \cdot \nabla u - \lambda u = 0$ in \mathbb{R}^d , then $u \equiv 0$. Also, the same statement is true with \tilde{L} in place of L .*

Proof. Suppose that u is not identically zero. Without loss of generality, assume $\sup_{\mathbb{R}^d} u > 0$ (otherwise consider $-u$). Since u goes to zero as $|x| \rightarrow \infty$, there exists $x_0 \in \mathbb{R}^d$ such that $u(x_0) = \sup_{\mathbb{R}^d} u$. Thus $\nabla u(x_0) = 0$ and

$$Lu(x_0) = \int_{\mathbb{R}^d} (u(x_0 + y) - u(x_0) - y \cdot \nabla u(x_0) \chi(y)) a(y) J(y) dy \leq 0.$$

Therefore we reach the contradiction. Indeed,

$$Lu(x_0) + b(x_0) \cdot \nabla u(x_0) - \lambda u(x_0) < 0.$$

The proof for \tilde{L} is almost identical. The lemma is proved. \square

This maximum principle yields the denseness of $(L + b \cdot \nabla - \lambda)C_0^\infty$ and $(\tilde{L} + b \cdot \nabla - \lambda)C_0^\infty$ in L_p .

Lemma 2.6. *Let $\lambda > 0$ and $b \in \mathbb{R}^d$ be independent of x . Then $(L + b \cdot \nabla - \lambda)C_0^\infty := \{Lu + b \cdot \nabla u - \lambda u : u \in C_0^\infty\}$ is dense in L_p for any $p \in (1, \infty)$. Also, the same statement holds with \tilde{L} in place of L .*

Proof. Due to the similarity we only prove the first statement. Suppose that the statement is false. Then by the Hahn-Banach theorem and Riesz's representation theorem, there exists a nonzero $v \in L_{p/(p-1)}$ such that

$$\int_{\mathbb{R}^d} (Lu(x) + b \cdot \nabla u(x) - \lambda u(x)) v(x) dx = 0 \quad (2.14)$$

for all $u \in C_0^\infty$.

Fixing $y \in \mathbb{R}^d$, we apply (2.14) with $u(y - \cdot)$. Then, due to Fubini's Theorem,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} (L^*u(y - x) - b \cdot \nabla u(y - x) - \lambda u(y - x)) v(x) dx \\ &= L^*u * v(y) - b \cdot (\nabla u * v(y)) - \lambda u * v(y) = (L^* - b \cdot \nabla - \lambda)(u * v)(y). \end{aligned}$$

Therefore from the previous lemma, we have $u * v = 0$ for any $u \in C_0^\infty$. Therefore, $v = 0$ (a.e.) and we have a contradiction. \square

Corollary 2.7 (Uniqueness). *Let $\lambda > 0$. Suppose that there exist $u, v \in \mathcal{H}_p^A$ satisfying*

$$Lu - \lambda u = 0, \quad \tilde{L}v - \lambda v = 0.$$

Then $u = v = 0$.

Proof. By the definition of a solution and the assumption of this corollary, there exists a sequence $\{u_n \in C_0^\infty\}$ such that for all $w \in C_0^\infty$

$$0 = \int_{\mathbb{R}^d} \lim_{n \rightarrow \infty} (Lu_n - \lambda u_n)w \, dx = \int_{\mathbb{R}^d} u(L^*w - \lambda w) \, dx.$$

Since $\{L^*w - \lambda w : w \in C_0^\infty\}$ is dense in $L_{p/(p-1)}$ owing to Lemma 2.6, we conclude $u = 0$, and by the same argument we have $v = 0$. \square

Here is our L_2 -theory. We emphasize that only (2.8) and (2.9) are assumed for the L_2 -theory. The proof of Theorem 2.8 is given in Section 3.

Theorem 2.8. *Let $\lambda > 0$. Then for any $f \in L_2$ there exist unique solutions $u, v \in \mathcal{H}_2^A$ of equation (2.12) and (2.13) respectively, and for these solutions we have*

$$\|\mathcal{A}u\|_{L_2} + \lambda\|u\|_{L_2} \leq N(d, \nu, \Lambda)\|f\|_{L_2}, \quad (2.15)$$

$$\|\mathcal{A}v\|_{L_2} + \lambda\|v\|_{L_2} \leq N(d, \nu, \Lambda)\|f\|_{L_2}. \quad (2.16)$$

The issue regarding the continuity of L (or \tilde{L}) : $\mathcal{H}_p^A \rightarrow L_p$ will be discussed later.

For the case $p \neq 2$, we consider the following conditions on $J(y) = j(|y|)$:

(H1): There exist constants $\kappa_1 > 0$ and $\alpha_0 > 0$ such that

$$j(t) \leq \kappa_1(s/t)^{d+\alpha_0}j(s), \quad \forall 0 < s \leq t. \quad (2.17)$$

Moreover, $\alpha_0 \leq 1$ if $\sigma \leq 1$ and $1 < \alpha_0 < 2$ if $\sigma > 1$.

(H2): There exists a constant $\kappa_2 > 0$ such that for all $t > 0$,

$$\int_{|y| \leq 1} |y|j(t|y|) \, dy \leq \kappa_2 j(t) \quad \text{if } \sigma \in (0, 1), \quad (2.18)$$

$$\int_{|y| \leq 1} |y|^2 j(t|y|) \, dz \leq \kappa_2 j(t) \quad \text{if } \sigma \geq 1. \quad (2.19)$$

Remark 2.9. (i) By taking $t = 1$ in (2.17),

$$j(1)\kappa_1^{-1}s^{-d-\alpha_0} \leq j(s), \quad \forall s \in (0, 1). \quad (2.20)$$

An upper bound of $j(s)$ near $s = 0$ is obtained in the following lemma.

(ii) **H1** and **H2** are needed even to guarantee the continuity of the operator $L : \mathcal{H}_2^A \rightarrow L_2$ (see Lemma 3.1).

Lemma 2.10. *Suppose*

$$j(s) \geq Cj(t), \quad \forall s \leq t, \quad (2.21)$$

*and **H2** hold. Then there exists a constant $N(d, \kappa_2, C) > 0$ such that for all $0 < s \leq t$*

$$j(t) \geq N(s/t)^{d+1}j(s) \quad (\text{if } \sigma < 1), \quad j(t) \geq N(s/t)^{d+2}j(s) \quad (\text{if } \sigma \geq 1). \quad (2.22)$$

On the other hand, if there exists $\alpha > 0$ so that $\alpha < 1$ if $\sigma < 1$, $\alpha < 2$ if $\sigma \geq 1$, and

$$j(t) \geq N(s/t)^{d+\alpha}j(s), \quad \forall 0 < s \leq t, \quad (2.23)$$

*then **H2** holds.*

Remark 2.11. By Lemma 2.10, both **H1** and **H2** hold if $0 < \alpha_0 \leq \alpha$ and

$$N^{-1}(s/t)^{d+\alpha}j(s) \leq j(t) \leq N(s/t)^{d+\alpha_0}j(s), \quad \forall 0 < s \leq t.$$

Example 2.12. Let $J(y) = j(|y|)$ be defined as in (1.4), that is for a Bernstein function $\phi(\lambda) = \int_{\mathbb{R}} (1 - e^{-\lambda t}) \mu(dt)$ and $u \in C_0^2$,

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(dt),$$

and

$$\begin{aligned} \phi(\Delta)u &= \int_{\mathbb{R}^d} (u(x+y) - u(x) - \nabla u(x) \cdot y I_{|y| \leq 1}) J(y) dy \\ &= -\mathcal{F}(\phi(|\xi|^2) \mathcal{F}(u)(\xi)). \end{aligned}$$

Then, **H1** and **H2** are satisfied if ϕ is given, for instance, by any one of

- (1) $\phi(\lambda) = \sum_{i=1}^n \lambda^{\alpha_i}$, $0 < \alpha_i < 1$;
- (2) $\phi(\lambda) = (\lambda + \lambda^\alpha)^\beta$, $\alpha, \beta \in (0, 1)$;
- (3) $\phi(\lambda) = \lambda^\alpha (\log(1 + \lambda))^\beta$, $\alpha \in (0, 1)$, $\beta \in (0, 1 - \alpha)$;
- (4) $\phi(\lambda) = \lambda^\alpha (\log(1 + \lambda))^{-\beta}$, $\alpha \in (0, 1)$, $\beta \in (0, \alpha)$;
- (5) $\phi(\lambda) = (\log(\cosh(\sqrt{\lambda})))^\alpha$, $\alpha \in (0, 1)$;
- (6) $\phi(\lambda) = (\log(\sinh(\sqrt{\lambda})) - \log \sqrt{\lambda})^\alpha$, $\alpha \in (0, 1)$.

This is because all these functions satisfy the conditions

$$\mathbf{A}: \exists 0 < \delta_1 \leq \delta_2 < 1,$$

$$N^{-1} \lambda^{\delta_1} \phi(t) \leq \phi(\lambda t) \leq N \lambda^{\delta_2} \phi(t), \quad \forall \lambda \geq 1, t \geq 1$$

$$\mathbf{B}: \exists 0 < \delta_3 \leq \delta_4 < 1,$$

$$N^{-1} \lambda^{\delta_3} \phi(t) \leq \phi(\lambda t) \leq N \lambda^{\delta_4} \phi(t), \quad \forall \lambda \leq 1, t \leq 1,$$

and under these condition one can prove (see [10])

$$N^{-1} \left(\frac{R}{r} \right)^{\delta_1 \wedge \delta_3} \leq \frac{\phi(R)}{\phi(r)} \leq N \left(\frac{R}{r} \right)^{\delta_2 \vee \delta_4}$$

and

$$N^{-1} \phi(|y|^{-2}) |y|^{-d} \leq J(y) \leq N \phi(|y|^{-2}) |y|^{-d}, \quad (2.24)$$

and consequently our conditions **H1** and **H2** hold. One can easily construct concrete examples of $j(r)$ using (2.24) and (1)-(6) (just replace λ by r^{-2}). See the tables at the end of [13] for more examples satisfying **A** and **B**.

Remark 2.13. If $p \neq 2$, our L_p -theory does not cover the case when the jump function $J(y)$ is related to the relativistic α -stable process with mass $m > 0$ (i.e. a subordinate Brownian motion with the infinitesimal generator $\phi(\Delta) = m - (m^{2/\alpha} - \Delta)^{\alpha/2}$). This is because the related jump function decreases exponentially fast at the infinity (for instance, see [7]) and thus condition **H2** fails (see (2.22)).

Proof of Lemma 2.10. Assume (2.21) and **H2** hold. We put $B_1 = \cup_{n=0}^{\infty} B_{(n)}$, where $B_{(n)} = B_{2^{-n}} \setminus B_{2^{-(n+1)}}$. Due to (2.21) for each $n \geq 0$,

$$\begin{aligned} \kappa_2 j(t) &\geq \int_{|y| \leq 1} |y|^2 j(t|y|) dy = \sum_{n=0}^{\infty} \int_{B_{(n)}} |y|^2 j(t|y|) dy \\ &\geq N \sum_{n=0}^{\infty} 2^{-(n+1)(d+2)} j(t2^{-n}) \geq N 2^{-(n+1)(d+2)} j(t2^{-n}). \end{aligned}$$

Put $s = t\lambda$, where $\lambda \in (0, 1)$, and take an integer $m(\lambda) \geq 0$ such that $2^{-(m+1)} \leq \lambda \leq 2^{-m}$. Then by (2.21),

$$j(t) \geq N 2^{-(m+2)(d+2)} j(2^{-(m+1)} t) \geq N \lambda^{d+2} j(\lambda t).$$

Similarly, $j(\lambda t) \leq \lambda^{-d-1} j(t)$ if $\sigma < 1$.

For the other direction, put $s = t|y|$ in (2.23). If $\sigma < 1$ then

$$\begin{aligned} \int_{|y| \leq 1} |y| j(t|y|) dy &\leq N j(t) \int_{|y| \leq 1} |y| \frac{j(t|y|)}{j(t)} dy \\ &\leq N j(t) \int_{|y| \leq 1} |y|^{-d-\alpha_1+1} dy \leq N j(t) \end{aligned}$$

and otherwise, that is, if $\sigma \geq 1$ then

$$\begin{aligned} \int_{|y| \leq 1} |y|^2 j(t|y|) dy &\leq N j(t) \int_{|y| \leq 1} |y|^2 \frac{j(t|y|)}{j(t)} dy \\ &\leq N j(t) \int_{|y| \leq 1} |y|^{-d-\alpha_2+2} dy \leq N j(t). \end{aligned}$$

The lemma is proved. \square

Define

$$\Psi(\xi) := - \int_{\mathbb{R}^d} (e^{i\xi \cdot y} - 1 - i(y \cdot \xi) \chi(y)) J(y) dy = \int_{\mathbb{R}^d} (1 - \cos \xi \cdot y) J(y) dy.$$

Then

$$\mathcal{A}u = \mathcal{F}^{-1}(-\Psi(\xi)\mathcal{F}u), \quad \forall u \in C_0^\infty.$$

By abusing the notation, we also use $\Psi(|\xi|)$ instead of $\Psi(\xi)$ because $\Psi(\xi)$ is rotationally invariant.

The following result will be used to prove the continuity of the operator L .

Lemma 2.14. *Suppose that (2.21) holds. Then there exists a constant $N(d, C) > 0$ such that for all $\xi \in \mathbb{R}^d$*

$$j(|\xi|) \leq N |\xi|^{-d} \Psi(|\xi|^{-1}). \quad (2.25)$$

Proof. By (2.21),

$$\begin{aligned} \Psi(|\xi|^{-1}) &= \int_{\mathbb{R}^d} (1 - \cos(y^1/|\xi|)) J(y) dy = |\xi|^d \int_{\mathbb{R}^d} (1 - \cos(y^1)) J(|\xi|y) dy \\ &\geq |\xi|^d \int_{|y| \leq 1} (1 - \cos(y^1)) J(|\xi|y) dy \\ &\geq C j(|\xi|) |\xi|^d \int_{|y| \leq 1} (1 - \cos(y^1)) dy \geq N j(|\xi|) |\xi|^d. \end{aligned}$$

Hence the lemma is proved. \square

The following condition will be considered for the case $\sigma = 1$. This condition is needed even to prove the continuity of L .

Assumption 2.15. If $\sigma = 1$ then

$$\int_{\partial B_r} y^i a(y) J(y) dS_r(y) = 0, \quad \forall r \in (0, \infty), i = 1, \dots, d, \quad (2.26)$$

where dS_r is the surface measure on ∂B_r .

Here is our L_p -theory for equation (2.27) below.

Theorem 2.16. Suppose that **H1** and **H2** hold and Assumption 2.15 also holds if $\sigma = 1$. Let $\lambda > 0$ and $p > 1$. Then for any $f \in L_p$ there exists a unique solution $u \in \mathcal{H}_p^A$ of the equation

$$Lu - \lambda u = f, \quad (2.27)$$

and for this solution we have

$$\|Au\|_{L_p} + \lambda \|u\|_{L_p} \leq N(d, p, \nu, \Lambda, J) \|f\|_{L_p}. \quad (2.28)$$

Moreover, L is a continuous operator from \mathcal{H}_p^A to L_p , and (2.28) holds for all $u \in \mathcal{H}_p^A$ with $f := Lu - \lambda u$.

The proof of this theorem will be given in Section 6.

Remark 2.17. Since the constant N in (2.28) does not depend on λ , for any $u \in \mathcal{H}_p^A$

$$\|Au\|_{L_p} \leq N \|Lu\|_{L_p}.$$

To study the equations with the operator \tilde{L} , we consider an additional condition, which always holds when $\sigma = 1$.

Assumption 2.18 (H3). Any one of the following (i)-(iv) holds:

(i) \mathcal{A} is a higher order differential operator than $I_{\sigma \neq 1} \nabla u$, that is for any $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ so that for any $u \in C_0^\infty$

$$I_{\sigma \neq 1} \|\nabla u\|_p \leq \varepsilon \|\mathcal{A}u\|_p + N(\varepsilon) \|u\|_p. \quad (2.29)$$

(ii) $\sigma < 1$ and

$$\int_{r \leq |y| \leq 1} y^i \left(a(y) - [a(y) \wedge a(-y)] \right) J(y) dy = 0, \quad \forall r \in (0, 1), i = 1, \dots, d. \quad (2.30)$$

(iii) $\sigma < 1$ and there exists a constant $\kappa_3 > 0$ such that for all $0 < t < 1$,

$$\int_{|z| \geq 1} |z| j(t|z|) dz \leq \kappa_2 j(t). \quad (2.31)$$

(iv) $\sigma > 1$ and

$$\int_{1 \leq |y| \leq r} y^i \left(a(y) - [a(y) \wedge a(-y)] \right) J(y) dy = 0, \quad \forall r > 1, i = 1, \dots, d. \quad (2.32)$$

Remark 2.19. (i) Note that (2.29) is satisfied if for some $\alpha > 1$,

$$\|\Delta^{\alpha/2}u\|_p \leq N(\|u\|_p + \|\mathcal{A}u\|_p), \quad \forall u \in C_0^\infty, \quad (2.33)$$

or, equivalently $|\xi|^\alpha(1 + \Psi(\xi))^{-1}$ is a L_p -Fourier multiplier. Thus, certain differentiability of $J(y)$ is required (see Lemma 2.20 below).

(ii) It is easy to check that (2.31) holds if for a $\alpha > 1$,

$$j(\lambda t) \leq N\lambda^{-d-\alpha}j(t), \quad \forall \lambda \in (1, \infty), \quad 0 < t < 1. \quad (2.34)$$

(iii) Obviously, (2.30) holds if $a(y) = a(-y)$ for $|y| \leq 1$, and (2.32) holds if $a(y) = a(-y)$ for $|y| \geq 1$.

Below we give a sufficient condition for (2.29).

Lemma 2.20. (i) **H3**-(i) holds if $\mathcal{A} = \phi(\Delta)$ for some Bernstein function ϕ satisfying

$$1 + \phi(|\xi|^2) \geq N|\xi|^\alpha, \quad \forall \xi \in \mathbb{R}^d, \quad (2.35)$$

where $\alpha > 1$ and $N > 0$.

(ii) All of **H1**, **H2** and **H3** hold if $\sigma > 1$, $\mathcal{A} = \phi(\Delta)$ and ϕ satisfies conditions **A** and **B** described in Example 2.12.

Proof. (i). Let $\phi(\lambda) = \int_{\mathbb{R}} (1 - e^{-\lambda t})\mu(dt)$, where $\int_{\mathbb{R}} (1 \wedge |t|)\mu(dt) < \infty$. Then from $t^n e^{-t} \leq N(n)(1 - e^{-t})$, we get

$$|\lambda|^n |D^n \phi(\lambda)| \leq N\phi(\lambda). \quad (2.36)$$

For any $u \in C_0^\infty$,

$$\mathcal{A}u = \mathcal{F}^{-1}(\phi(|\xi|^2)\mathcal{F}(u)(\xi)),$$

$$\Delta^{\alpha/2}u = \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}(u)(\xi)) = \mathcal{F}^{-1}(\eta(\xi)(1 + \phi(|\xi|^2)\mathcal{F}(u)(\xi)),$$

where $\eta(\xi) = |\xi|^\alpha(1 + \phi(|\xi|^2))^{-1}$. Using (2.35) and (2.36), one can easily check

$$|D^n \eta(\xi)| \leq N(n)|\xi|^{-n}, \quad \forall \xi,$$

and therefore η is a Fourier multiplier (see Theorem IV.3.2 of [16]) and

$$\|\Delta^{\alpha/2}u\| \leq N(\|u\|_p + \|\mathcal{A}u\|_p),$$

$$\|\nabla u\|_p \leq \varepsilon \|\Delta^{\alpha/2}u\|_p + N(\varepsilon)\|u\|_p \leq N\varepsilon \|\mathcal{A}u\|_p + N\|u\|_p.$$

(ii) If **A** and **B** hold, then as explained before both **H1**, **H2** hold, and we also have (see (2.24)),

$$N^{-1}\phi(|y|^{-2})|y|^{-d} \leq J(y) \leq N\phi(|y|^{-2})|y|^{-d}.$$

Thus if $|\xi| \geq 1$, then

$$\phi(|\xi|^2) \geq N|\xi|^{-d}J(|\xi|^{-1}) \geq N|\xi|^{\alpha_0},$$

where (2.20) is used for the last inequality. Hence the lemma is proved. \square

Here is our L_p -theory for equation (2.37) below.

Theorem 2.21. *Suppose that **H1**, **H2** and **H3** hold and Assumption 2.15 also holds if $\sigma = 1$. Let $\lambda > 0$ and $p > 1$. Then for any $f \in L_p$ there exists a unique solution $u \in \mathcal{H}_p^A$ of the equation*

$$\tilde{L}u - \lambda u = f, \quad (2.37)$$

and for this solution we have

$$\|\mathcal{A}u\|_{L_p} + \lambda\|u\|_{L_p} \leq N(d, \nu, \Lambda, \lambda, J)\|f\|_{L_p}. \quad (2.38)$$

The proof of this theorem will be given in Section 6. Actually the constant N in (2.38) is independent of λ except the case when **H3**(i) is assumed.

3. L_2 -THEORY

In this section we prove (2.15) and (2.16). These estimates and Lemma 2.6 yield the unique solvability of equations (2.12) and (2.13). The Fourier transform and Parseval's identity are used to prove these estimates.

Lemma 3.1. *Let $\lambda \geq 0$ be a constant.*

(i) *For any $u \in C_0^\infty$*

$$\|\mathcal{A}u\|_{L_2} + \lambda\|u\|_{L_2} \leq N(d, \nu)\|Lu - \lambda u\|_{L_2} \quad (3.39)$$

and

$$\|\mathcal{A}u\|_{L_2} + \lambda\|u\|_{L_2} \leq N(d, \nu)\|\tilde{L}u - \lambda u\|_{L_2}. \quad (3.40)$$

(ii) *Let **H1** hold and $\sigma > 1$. Then both L and \tilde{L} are continuous operators from \mathcal{H}_2^A to L_2 , and for any $u \in C_0^\infty$,*

$$\|Lu\|_{L_2} \leq N\|\mathcal{A}u\|_{L_2}, \quad \|\tilde{L}u\|_{L_2} \leq N\|u\|_{\mathcal{H}_2^A}, \quad (3.41)$$

where $N = N(d, \nu, J)$. Moreover, (3.39) and (3.40) hold for any $u \in \mathcal{H}_2^A$.

(iii) *Let **H1** and **H2** hold, and Assumption 2.15 also hold if $\sigma = 1$. Then the claims of (ii) hold for L (not for \tilde{L}) for any $\sigma \in (0, 1]$.*

Proof. (i). Let $u \in C_0^\infty$. Taking the Fourier transform, we get

$$\mathcal{F}(Lu)(\xi) = \mathcal{F}u(\xi) \int_{\mathbb{R}^d} (e^{i\xi \cdot y} - 1 - iy \cdot \xi \chi(y))a(y)J(y)dy. \quad (3.42)$$

By Parseval's identity,

$$\begin{aligned} \int_{\mathbb{R}^d} |Lu(x)|^2 dx &= (2\pi)^{-d} \int_{\mathbb{R}^d} |\mathcal{F}(Lu)(\xi)|^2 d\xi \\ &\geq (2\pi)^{-d} \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \operatorname{Re} \int_{\mathbb{R}^d} (e^{i\xi \cdot y} - 1 - iy \cdot \xi \chi(y))a(y)J(y)dy \right|^2 d\xi \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y))a(y)J(y)dy \right|^2 d\xi \\ &\geq (2\pi)^{-d} \nu^2 \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y))J(y)dy \right|^2 d\xi \\ &= \nu^2 \int_{\mathbb{R}^d} |\mathcal{A}u|^2 dx, \end{aligned}$$

where the facts that $1 - \cos(\xi \cdot y)$ is nonnegative and $a(y) \geq \nu$ are used above.

Similarly, since uLu is real,

$$\begin{aligned}
& - \int_{\mathbb{R}^d} uLu \, dx = -(2\pi)^{-d} \int_{\mathbb{R}^d} \mathcal{F}(Lu)(\xi) \overline{\mathcal{F}(u)(\xi)} \, d\xi \\
& = -(2\pi)^{-d} \int_{\mathbb{R}^d} |\mathcal{F}(u)(\xi)|^2 \operatorname{Re} \int_{\mathbb{R}^d} \left(e^{i\xi \cdot y} - 1 - iy \cdot \xi \chi^{(\sigma)}(y) \right) a(y) J(y) \, dy d\xi \\
& = (2\pi)^{-d} \int_{\mathbb{R}^d} |\mathcal{F}(u)(\xi)|^2 \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) a(y) J(y) \, dy d\xi \\
& \geq \frac{\nu}{2} (2\pi)^{-d} \int_{\mathbb{R}^d} |\mathcal{F}(u)(\xi)|^2 \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) J(y) \, dy d\xi \\
& = -\frac{\nu}{2} \int_{\mathbb{R}^d} u \mathcal{A} u \, dx.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \int_{\mathbb{R}^d} |Lu - \lambda u|^2 \, dx \\
& = \int_{\mathbb{R}^d} |Lu|^2 \, dx - 2\lambda \int_{\mathbb{R}^d} uLu \, dx + \lambda^2 \int_{\mathbb{R}^d} |u|^2 \, dx \\
& \geq \nu^2 \int_{\mathbb{R}^d} |\mathcal{A}u|^2 \, dx - \lambda\nu \int_{\mathbb{R}^d} u \mathcal{A} u \, dx + \lambda^2 \int_{\mathbb{R}^d} |u|^2 \, dx \\
& \geq \nu^2 \int_{\mathbb{R}^d} |\mathcal{A}u|^2 \, dx - \frac{\nu^2}{2} \int_{\mathbb{R}^d} u^2 \, dx - \frac{\lambda^2}{2} \int_{\mathbb{R}^d} |\mathcal{A}u|^2 \, dx + \lambda^2 \int_{\mathbb{R}^d} |u|^2 \, dx \\
& = \frac{\nu^2}{2} \int_{\mathbb{R}^d} |\mathcal{A}u|^2 \, dx + \frac{\lambda^2}{2} \int_{\mathbb{R}^d} |u|^2 \, dx.
\end{aligned}$$

Thus (3.39) holds. Also, (3.40) is proved similarly.

(ii)-(iii). Next, we prove (3.41) for any $u \in C_0^\infty$. Unlike the case $j(r) = r^{-d-\alpha}$, the proof is not completely trivial. Condition **H1** is needed if $\sigma > 1$, and **H2** is additionally needed if $\sigma \leq 1$.

By using (3.42) and Parseval's identity again,

$$\begin{aligned}
& \int_{\mathbb{R}^d} |Lu(x)|^2 dx = (2\pi)^{-d} \int_{\mathbb{R}^d} |\mathcal{F}(Lu)(\xi)|^2 d\xi \\
& = (2\pi)^{-d} \left[\int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \operatorname{Re} \int_{\mathbb{R}^d} (e^{i\xi \cdot y} - 1 - iy \cdot \xi \chi(y)) a(y) J(y) \, dy \right|^2 d\xi \right. \\
& \quad \left. + \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \operatorname{Im} \int_{\mathbb{R}^d} (e^{i\xi \cdot y} - 1 - iy \cdot \xi \chi(y)) a(y) J(y) \, dy \right|^2 d\xi \right] \\
& \leq (2\pi)^{-d} \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) a(y) J(y) \, dy \right|^2 d\xi \\
& \quad + (2\pi)^{-d} \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y||\xi| \geq 1} (\sin(\xi \cdot y) - y \cdot \xi \chi(y)) a(y) J(y) \, dy \right|^2 d\xi \\
& \quad + (2\pi)^{-d} \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y||\xi| < 1} (\sin(\xi \cdot y) - y \cdot \xi \chi(y)) a(y) J(y) \, dy \right|^2 d\xi \\
& := \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^d} |\tilde{L}u|^2 dx = \tilde{\mathcal{I}}_1 + \tilde{\mathcal{I}}_2 + \tilde{\mathcal{I}}_3,$$

where $\tilde{\mathcal{I}}_i$ are obtained by replacing $\chi(y)$ in \mathcal{I}_i with $I_{B_1}(y)$. Here \mathcal{I}_1 and $\tilde{\mathcal{I}}_1$ are easily controlled by $N\|\mathcal{A}u\|_{L_2}^2$.

Due to **H1**, (2.26), the definition of χ , and the change of variables $y \rightarrow \frac{y}{|\xi|}$,

$$\begin{aligned} \mathcal{I}_2 &\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 |\xi|^{-2d} \left| \int_{|y| \geq 1} \left(\sin\left(\frac{\xi}{|\xi|} \cdot y\right) - y \cdot \frac{\xi}{|\xi|} \chi\left(\frac{y}{|\xi|}\right) a\left(\frac{y}{|\xi|}\right) J\left(\frac{y}{|\xi|}\right) dy \right)^2 d\xi \right. \\ &\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 |\xi|^{-2d} j(1/|\xi|)^2 \\ &\quad \times \left(\int_{|y| \geq 1} \left| \sin\left(\frac{\xi}{|\xi|} \cdot y\right) - I_{\sigma \neq 1} y \cdot \frac{\xi}{|\xi|} \chi\left(\frac{y}{|\xi|}\right) a\left(\frac{y}{|\xi|}\right) |y|^{-d-\alpha_0} dy \right|^2 d\xi \right) \\ &\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 |\xi|^{-2d} j(1/|\xi|)^2 d\xi. \end{aligned}$$

Hence, by Lemma 2.14,

$$\mathcal{I}_2 \leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 (\Psi(\xi))^2 d\xi = N \int_{\mathbb{R}^d} |\mathcal{A}u|^2 dx.$$

Similarly, if $\sigma > 1$,

$$\begin{aligned} \tilde{\mathcal{I}}_2 &\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 |\xi|^{-2d} j(1/|\xi|)^2 \\ &\quad \times \left(\int_{|y| \geq 1} \left| \sin\left(\frac{\xi}{|\xi|} \cdot y\right) - I_{\sigma > 1} y \cdot \frac{\xi}{|\xi|} I_{|y| \leq |\xi|} \left| a\left(\frac{y}{|\xi|}\right) |y|^{-d-\alpha_0} dy \right|^2 d\xi \right) \\ &\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 |\xi|^{-2d} j(1/|\xi|)^2 d\xi \leq N \int_{\mathbb{R}^d} |\mathcal{A}u|^2 dx. \end{aligned}$$

Also, using the fundamental theorem of calculus, the definition of χ and (2.26),

$$\begin{aligned} \mathcal{I}_3 &\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y||\xi| < 1} (\sin(\xi \cdot y) - y \cdot \xi \chi(y)) a(y) J(y) dy \right|^2 d\xi \\ &= N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y||\xi| < 1} \int_0^1 \frac{d}{dt} (\sin(t\xi \cdot y) - ty \cdot \xi \chi(y)) dt a(y) J(y) dy \right|^2 d\xi \\ &= N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y||\xi| < 1} (\xi \cdot y) \int_0^1 (\cos(t\xi \cdot y) - \chi(y)) dt a(y) J(y) dy \right|^2 d\xi \\ &= I_{\sigma \leq 1} N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y||\xi| < 1} (\xi \cdot y) \int_0^1 \cos(t\xi \cdot y) dt a(y) J(y) dy \right|^2 d\xi \\ &\quad + I_{\sigma > 1} N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y||\xi| < 1} (\xi \cdot y) \int_0^1 (\cos(t\xi \cdot y) - 1) dt a(y) J(y) dy \right|^2 d\xi. \end{aligned}$$

Observe that by **H1**, for any $t \in (0, 1)$,

$$\Psi(t|\xi|) = \int_{\mathbb{R}^d} (1 - \cos(ty \cdot \xi)) J(y) dy = t^{-d} \int_{\mathbb{R}^d} (1 - \cos(y \cdot \xi)) J(t^{-1}y) dy \leq N t^{\alpha_0} \Psi(|\xi|).$$

Thus, if $\sigma > 1$,

$$\mathcal{I}_3 \leq N \int_{\mathbb{R}^d} |\mathcal{F}u|^2 \left(\int_0^1 \Psi(t|\xi|) dt \right)^2 d\xi \leq N \|\mathcal{A}u\|_{L_2}^2.$$

Also, if $\sigma > 1$,

$$\begin{aligned} \tilde{\mathcal{I}}_3 &\leq (2\pi)^{-d} \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y||\xi| < 1} (\xi \cdot y) \int_0^1 \cos(t\xi \cdot y) I_{|y| \geq 1} dt a(y) J(y) dy \right|^2 d\xi \\ &\quad + (2\pi)^{-d} \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| \int_{|y||\xi| < 1} (\xi \cdot y) \int_0^1 (1 - \cos(t\xi \cdot y)) dt a(y) J(y) dy \right|^2 d\xi \\ &\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left(\int_{|y| \geq 1} J(y) dy \right)^2 d\xi + N \int_{\mathbb{R}^d} |\mathcal{F}u|^2 \left(\int_0^1 \Psi(t|\xi|) dt \right)^2 d\xi \\ &\leq N \|u\|_{\mathcal{H}_2^A}^2. \end{aligned}$$

Thus (3.41) is proved if $\sigma > 1$, and (3.39) and (3.40) are obtained for general $u \in \mathcal{H}_2^A$ owing to (3.41). Therefore (ii) is proved.

Now assume $\sigma \leq 1$. To estimate \mathcal{I}_3 we use the Fubini's Theorem, the change of variable $|\xi|ty \rightarrow y$, **H1**, **H2**, and Lemma 2.14

$$\begin{aligned} \mathcal{I}_3 &\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \\ &\quad \times \left| \int_0^1 t^{-d-1} |\xi|^{-d} \int_{|y| < t} \left(\frac{\xi}{|\xi|} \cdot y \right) \cos\left(\frac{\xi}{|\xi|} \cdot y\right) a\left(\frac{y}{|\xi|t}\right) J\left(\frac{y}{|\xi|t}\right) dy dt \right|^2 d\xi \\ &\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| |\xi|^{-d} \int_0^1 t^{-d-1} \int_{|y| < 1} |y| J\left(\frac{y}{|\xi|t}\right) dy dt \right|^2 d\xi \\ &\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| |\xi|^{-d} \int_0^1 t^{\alpha_0-1} dt \int_{|y| < 1} |y| J(y/|\xi|) dy \right|^2 d\xi \\ &\leq N \int_{\mathbb{R}^d} |\mathcal{F}u(\xi)|^2 \left| |\xi|^{-d} j(1/|\xi|) \right|^2 d\xi \leq N \|\mathcal{A}u\|_{L_2}^2. \end{aligned}$$

Therefore the lemma is proved. \square

Corollary 2.7 and Lemmas 2.6 and 3.1 easily prove Theorem 2.8.

4. SOME HÖLDER ESTIMATES

In this section obtain some Hölder estimates for functions $u \in \mathcal{H}_2^A \cap C_b^\infty$. The estimates will be used later for the estimates of the mean oscillation. Throughout this section we assume Assumption 2.15 holds if $\sigma = 1$.

Lemma 4.1. *For any $\alpha \in (0, 1)$, $b \in \mathbb{R}^d$, and a nonnegative measurable function $\mathcal{K}(z)$, there exist $\eta_1, \eta_2 \in (0, 1/4)$, depending only on α , such that*

$$\begin{aligned} & \int_{\mathcal{C}} [(|b + 2z|^\alpha + |b - 2z|^\alpha - 2|b|^\alpha) \mathcal{K}(z)] dz \\ & \leq -2^{\alpha-3} \alpha (1 - \alpha) \int_{\mathcal{C}} |b|^{\alpha-2} |z|^2 \mathcal{K}(z) dz, \end{aligned} \quad (4.43)$$

where

$$\mathcal{C} = \{|z| < \eta_1 |b| : |z \cdot b| \geq (1 - \eta_2) |b| |z|\}.$$

Proof. We repeat the proof of Lemma 4.2 in [9] with few minor changes. Put $\eta(t) := b + 2tz$ and $\varphi(t) := |b + 2tz|^\alpha = |\eta(t)|^\alpha$ for $z \in \mathcal{C}$. Then

$$\begin{aligned} \varphi''(t) &= \sum_{i,j=1}^d (\alpha(\alpha-2)(\eta_i(t))(\eta_j(t))|\eta(t)|^{\alpha-4} + I_{i=j} \alpha |\eta(t)|^{\alpha-2}) 4z_i z_j \\ &= 4\alpha(\alpha-2)|\eta(t)|^{\alpha-4} |\eta(t) \cdot z|^2 + 4\alpha |\eta(t)|^{\alpha-2} |z|^2 \\ &= 4\alpha |b + 2tz|^{\alpha-4} [(\alpha-2)|(b + 2tz) \cdot z|^2 + |b + 2tz|^2 |z|^2]. \end{aligned}$$

For $t \in [-1, 1]$ and $z \in \mathcal{C}$, observe that,

$$|b + 2tz|^2 \leq (1 + 2\eta_1)^2 |b|^2$$

and

$$\begin{aligned} |(b + 2tz) \cdot z| &= |b \cdot z + 2t|z|^2| \geq |b \cdot z| - 2|z|^2 \\ &\geq (1 - \eta_2) |b| |z| - 2|z|^2 \geq (1 - 2\eta_1 - \eta_2) |z| |b|. \end{aligned}$$

Thus

$$\varphi''(t) \leq 4\alpha |a + 2tz|^{\alpha-4} [(\alpha-2)(1 - 2\eta_1 - \eta_2)^2 + (1 + 2\eta_1)^2] |b|^2 |z|^2. \quad (4.44)$$

Since $(1 - 2\eta_1 - \eta_2)^2 \rightarrow 1$ and $(1 + 2\eta_1)^2 \rightarrow 1$ as $\eta_1, \eta_2 \downarrow 0$, one can choose sufficiently small $\eta_1, \eta_2 \in (0, 1/4)$, depending only on $\alpha \in (0, 1)$, such that

$$(\alpha-2)(1 - 2\eta_1 - \eta_2)^2 + (1 + 2\eta_1)^2 \leq (\alpha-1)/2.$$

By combining this with (4.44)

$$\varphi''(t) \leq -2\alpha(1 - \alpha) |b + 2tz|^{\alpha-4} |b|^2 |z|^2. \quad (4.45)$$

Furthermore observe that

$$|b + 2tz|^{\alpha-4} \geq (1 + 2\eta_1)^{\alpha-4} |b|^{\alpha-4} \geq 2^{\alpha-4} |b|^{\alpha-4}.$$

Therefore, from (4.45)

$$\varphi''(t) \leq -2^{\alpha-3} \alpha (1 - \alpha) |b|^{\alpha-2} |z|^2, \quad t \in [-1, 1], \quad z \in \mathcal{C}.$$

In addition to this, to prove (4.43), it is enough to use the fact that there exists $t_0 \in (-1, 1)$ satisfying

$$\varphi(1) + \varphi(-1) - 2\varphi(0) = \varphi''(t_0),$$

which can be shown by the mean value theorem. The lemma is proved. \square

Theorem 4.2. *Let $R > 0, \lambda \geq 0$ and **H1** hold. Suppose $f \in L_\infty(B_1)$ and $u, \tilde{u} \in C_b^2(B_R) \cap L_1(\mathbb{R}^d, w_R)$, where $w_R(x) = \frac{1}{1/j(R)+1/J(x/2)}$. Also assume*

$$Lu - \lambda u = f, \quad \tilde{L}\tilde{u} - \lambda \tilde{u} = f \quad \text{in } B_R. \quad (4.46)$$

(i) *For any $\alpha \in (0, \min\{1, \alpha_0\})$ and $0 < r < R$, it holds that*

$$\begin{aligned} [u]_{C^\alpha(B_r)} &\leq N r_1^{-\alpha} \|u\|_{L_\infty(B_R)} \\ &\quad + N \frac{\|u\|_{L_\infty(B_R)}}{j(r_1) r_1^{d+\alpha}} \left(r_1^{-2} \int_{B_{r_1}} |z|^2 J(z) dz + I_{\sigma < 1} r_1^{-1} \int_{B_{r_1}} |z| J(z) dz \right) \\ &\quad + N \left(\frac{1}{r_1^{d+\alpha} j(R)} \|u\|_{L_1(\mathbb{R}^d, w_R)} + \frac{1}{j(r_1) r_1^{d+\alpha}} \text{osc}_{B_R} f \right), \end{aligned} \quad (4.47)$$

where $r_1 = (R - r)/2$ and $N = N(d, \nu, \Lambda, \kappa_1, \alpha_0, \alpha)$.

Consequently, if **H2** is additionally assumed, then

$$[u]_{C^\alpha(B_r)} \leq N \left(r_1^{-\alpha} \|u\|_{L_\infty(B_R)} + \frac{1}{r_1^{d+\alpha} j(R)} \|u\|_{L_1(\mathbb{R}^d, w_R)} + \frac{\text{osc}_{B_R} f}{j(r_1) r_1^{d+\alpha}} \right). \quad (4.48)$$

(ii) *In addition to **H1**, let one of **H3(ii)**- **H3(iv)** hold. Then (4.47) holds for \tilde{u} . Consequently, if **H2** additionally holds, (4.48) holds for \tilde{u} .*

Proof. We adopt the method used in [9] (cf. [3]). Assume that u is not identically zero in B_r . Set

$$r_1 = (R - r)/2, \quad r_2 = (R + r)/2, \quad w(t, x) = I_{B_R}(x) u(t, x).$$

For $x \in B_{r_2}$, $u(x) = v(x)$ and $\nabla u(x) = \nabla w(x)$. Thus

$$Lu(x) = Lw(t, x) + \int_{|z| \geq r_1} (u(t, x+z) - w(t, x+z)) a(z) J(z) dz.$$

So in B_{r_2}

$$Lw(x) - \lambda w = g(x) + f(x), \quad (4.49)$$

where

$$g(x) = - \int_{|z| \geq r_1} (u(x+z) - w(x+z)) a(z) J(z) dz.$$

Note that by **H1**

$$\|g\|_{L_\infty(B_R)} \leq N \frac{j(r_1)}{j(R)} \|u\|_{L_1(\mathbb{R}^d, w_R)}, \quad (4.50)$$

where $N = N(d, \Lambda)$. Indeed, this comes from the fact that for all $|z| \geq r_1$, $x \in B_R$, and $|x+z| \leq R$

$$|j(z)| \leq N j(r_1) \leq \frac{j(r_1)}{j(R)} \cdot \frac{N}{1/j(R) + 1/j(|x+z|/2)}.$$

For $x_0 \in B_r$ and $\alpha \in (0, \min\{1, \alpha_0\})$, we define

$$M(x, y) := w(x) - w(y) - C|x - y|^\alpha - 8r_1^{-2} \|u\|_{L_\infty(B_R)} |x - x_0|^2,$$

where C is a positive constant which will be chosen later so that it is independent of x_0 and

$$\sup_{x, y \in \mathbb{R}^d} M(x, y) \leq 0. \quad (4.51)$$

For $x \in \mathbb{R}^d \setminus B_{r_1/2}(x_0)$,

$$w(x) - w(y) \leq 2\|u\|_{L_\infty(B_R)} \leq 8r_1^{-2}\|u\|_{L_\infty(B_R)}|x - x_0|^2. \quad (4.52)$$

This shows

$$M(x, y) \leq 0, \quad x \in \mathbb{R}^d \setminus B_{r_1/2}(x_0).$$

Assume that there exist $x, y \in \mathbb{R}^d$ such that $M(x, y) > 0$. We will get the contradiction by choosing an appropriate constant C . Due to (4.52), $x \in B_{r_1/2}(x_0)$. Moreover

$$w(x) - w(y) > C|x - y|^\alpha,$$

which implies

$$|x - y|^\alpha < \frac{2\|u\|_{L_\infty(B_R)}}{C}. \quad (4.53)$$

If we take C large enough so that $C \geq 2(r_1/2)^{-\alpha}\|u\|_{L_\infty(B_R)}$, then

$$y \in B_{r+r_1}.$$

Therefore, there exist $\bar{x}, \bar{y} \in B_{r+r_1}$ satisfying

$$\sup_{x, y \in \mathbb{R}^d} M(x, y) = M(\bar{x}, \bar{y}) > 0.$$

Moreover, from (4.49)

$$\begin{aligned} -2\|g\|_{L_\infty(B_R)} - \text{osc}_{B_R} f &\leq (Lw(\bar{x}) - \lambda w(\bar{x})) - (Lw(\bar{y}) - \lambda w(\bar{y})) \\ &= (Lw(\bar{x}) - Lw(\bar{y})) + \lambda(w(\bar{y}) - w(\bar{x})) \\ &\leq Lw(\bar{x}) - Lw(\bar{y}) := \mathcal{I}. \end{aligned} \quad (4.54)$$

Put $K(z) := a(z)J(z)$ and

$$K_1(z) := K(z) \wedge K(-z), \quad K_2(z) := K(z) - K_1(z).$$

By L_1 and L_2 , respectively, we denote the operators with kernels K_1 and K_2 . Then

$$\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2,$$

where

$$\mathcal{I}_1 := L_1 w(\bar{x}) - L_1 w(\bar{y}) \quad \text{and} \quad \mathcal{I}_2 := L_2 w(\bar{x}) - L_2 w(\bar{y}).$$

Since K_1 is symmetric (i.e. $K_1(z) = K_1(-z)$),

$$\mathcal{I}_1 = \frac{1}{2} \int_{\mathbb{R}^d} \mathcal{J}(\bar{x}, \bar{y}, z) K_1(z) dz,$$

where

$$\mathcal{J}(\bar{x}, \bar{y}, z) = w(\bar{x} + z) + w(\bar{x} - z) - 2w(\bar{x}) - w(\bar{y} + z) - w(\bar{y} - z) + 2w(\bar{y}).$$

Also, since $M(x, y)$ attains its maximum at (\bar{x}, \bar{y}) ,

$$\begin{aligned} &w(\bar{x} + z) - w(\bar{y} + z) - C|\bar{x} - \bar{y}|^\alpha - 8r_1^{-2}\|u\|_{L_\infty(B_R)}|\bar{x} + z - x_0|^2 \\ &\leq w(\bar{x}) - w(\bar{y}) - C|\bar{x} - \bar{y}|^\alpha - 8r_1^{-2}\|u\|_{L_\infty(B_R)}|\bar{x} - x_0|^2 \end{aligned} \quad (4.55)$$

and

$$\begin{aligned} &w(\bar{x} - z) - w(\bar{y} - z) - C|\bar{x} - \bar{y}|^\alpha - 8r_1^{-2}\|u\|_{L_\infty(B_R)}|\bar{x} - z - x_0|^2 \\ &\leq w(\bar{x}) - w(\bar{y}) - C|\bar{x} - \bar{y}|^\alpha - 8r_1^{-2}\|u\|_{L_\infty(B_R)}|\bar{x} - x_0|^2 \end{aligned} \quad (4.56)$$

for all $z \in \mathbb{R}^d$. By combining these two inequalities,

$$\mathcal{J}(\bar{x}, \bar{y}, z) \leq 8r_1^{-2} \|u\|_{L_\infty(B_R)} (|\bar{x} + z - x_0|^2 + |\bar{x} - z - x_0|^2 - 2|\bar{x} - x_0|^2). \quad (4.57)$$

Similarly,

$$\begin{aligned} & w(\bar{x} + z) - w(\bar{y} - z) - C|\bar{x} - \bar{y} + 2z|^\alpha - 8r_1^{-2} \|u\|_{L_\infty(B_R)} |\bar{x} + z - x_0|^2 \\ & \leq w(\bar{x}) - w(\bar{y}) - C|\bar{x} - \bar{y}|^\alpha - 8r_1^{-2} \|u\|_{L_\infty(B_R)} |\bar{x} - x_0|^2, \\ & w(\bar{x} - z) - w(\bar{y} + z) - C|\bar{x} - \bar{y} - 2z|^\alpha - 8r_1^{-2} \|u\|_{L_\infty(B_R)} |\bar{x} - z - x_0|^2 \\ & \leq w(\bar{x}) - w(\bar{y}) - C|\bar{x} - \bar{y}|^\alpha - 8r_1^{-2} \|u\|_{L_\infty(B_R)} |\bar{x} - x_0|^2. \end{aligned}$$

It follows that, for any $z \in \mathbb{R}^d$,

$$\begin{aligned} \mathcal{J}(\bar{x}, \bar{y}, z) & \leq C(|\bar{x} - \bar{y} + 2z|^\alpha + |\bar{x} - \bar{y} - 2z|^\alpha - 2|\bar{x} - \bar{y}|^\alpha) \\ & \quad + 8r_1^{-2} \|u\|_{L_\infty(B_R)} (|\bar{x} + z - x_0|^2 + |\bar{x} - z - x_0|^2 - 2|\bar{x} - x_0|^2). \end{aligned} \quad (4.58)$$

Put $b = \bar{x} - \bar{y}$. Since (\bar{x}, \bar{y}) satisfy (4.53), $|b| < r_1/2$ if $C \geq 2(r_1/2)^{-\alpha} \|u\|_{L_\infty(B_R)}$. Also set for $\eta_1, \eta_2 \in (0, 1/4)$ specified in Lemma 4.1,

$$\mathcal{C} = \{|z| < \eta_1 |b| : |z \cdot b| \geq (1 - \eta_2) |b| |z|\}.$$

Then

$$\begin{aligned} 2\mathcal{I}_1 &= \int_{|z| \geq r_1/2} \mathcal{J}(\bar{x}, \bar{y}, z) K_1(z) dz + \int_{B_{r_1/2} \setminus \mathcal{C}} \mathcal{J}(\bar{x}, \bar{y}, z) K_1(z) dz \\ & \quad + \int_{\mathcal{C}} \mathcal{J}(\bar{x}, \bar{y}, z) K_1(z) dz := \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13}. \end{aligned} \quad (4.59)$$

Note that by **H1**,

$$\mathcal{I}_{11} \leq Nj(r_1/2)r_1^d \|u\|_{L_\infty(B_R)}.$$

Indeed,

$$\begin{aligned} \mathcal{I}_{11} & \leq N \|u\|_{L_\infty(B_R)} \int_{|z| \geq r_1/2} J(z) dz \\ & \leq Nr_1^d \|u\|_{L_\infty(B_R)} \int_{|z| \geq 1} J(r_1 z/2) dz \\ & \leq Nj(r_1/2)r_1^d \|u\|_{L_\infty(B_R)} \int_{|z| \geq 1} |z|^{-d-\alpha_0} dz. \end{aligned}$$

On the other hand from (4.57), it follows that

$$\begin{aligned} \mathcal{I}_{12} & \leq 8r_1^{-2} \|u\|_{L_\infty(B_R)} \int_{B_{r_1/2} \setminus \mathcal{C}} (|\bar{x} + z - x_0|^2 + |\bar{x} - z - x_0|^2 - 2|\bar{x} - x_0|^2) K_1(z) dz \\ & \leq Nr_1^{-2} \|u\|_{L_\infty(B_R)} \int_{B_{r_1/2}} |z|^2 J(z) dz. \end{aligned}$$

Next using (4.58) we obtain

$$\begin{aligned} \mathcal{I}_{13} & \leq C \int_{\mathcal{C}} (|\bar{x} - \bar{y} + 2z|^\alpha + |\bar{x} - \bar{y} - 2z|^\alpha - 2|\bar{x} - \bar{y}|^\alpha) K_1(z) dz \\ & \quad + 8r_1^{-2} \|u\|_{L_\infty(B_R)} \int_{\mathcal{C}} (|\bar{x} + z - x_0|^2 + |\bar{x} - z - x_0|^2 - 2|\bar{x} - x_0|^2) K_1(z) dz \\ & := \mathcal{I}_{131} + \mathcal{I}_{132}. \end{aligned}$$

The term \mathcal{I}_{132} is again bounded by

$$Nr_1^{-2}\|u\|_{L_\infty(B_R)} \int_{B_{r_1/2}} |z|^2 J(z) \, dz.$$

Furthermore, from lemma 4.1

$$\mathcal{I}_{131} \leq -2^{\alpha-3}C\alpha(1-\alpha) \int_{\mathcal{C}} |b|^{\alpha-2}|z|^2 K_1(z) dz.$$

Combining all these facts above, we obtain

$$\begin{aligned} \mathcal{I}_1 &\leq N\|u(\cdot)\|_{L_\infty(B_R)} \left(j(r_1/2)r_1^d + r_1^{-2} \int_{B_{r_1/2}} |z|^2 J(z) \, dz \right) \\ &\quad - 2^{\alpha-3}C\alpha(1-\alpha) \int_{\mathcal{C}} |b|^{\alpha-2}|z|^2 K_1(z) dz. \end{aligned} \quad (4.60)$$

For \mathcal{I}_2 , we first consider **the case** $\sigma < 1$. In this case,

$$\begin{aligned} \mathcal{I}_2 &= \int_{|z| \geq r_1/2} (w(\bar{x}+z) - w(\bar{x}) - w(\bar{y}+z) + w(\bar{y})) K_2(z) \, dz \\ &\quad + \int_{B_{r_1/2}} (w(\bar{x}+z) - w(\bar{x}) - w(\bar{y}+z) + w(\bar{y})) K_2(z) \, dz := \mathcal{I}_{21} + \mathcal{I}_{22}. \end{aligned}$$

Analogously to \mathcal{I}_{11} , we bound \mathcal{I}_{21} by $Nj(r_1/2)r_1^d\|u\|_{L_\infty(B_R)}$. For the other term \mathcal{I}_{22} , since $|\bar{x} - x_0| < r_1/2$, from (4.55)

$$\begin{aligned} \mathcal{I}_{22} &\leq Nr_1^{-2}\|u\|_{L_\infty(B_R)} \int_{B_{r_1/2}} (|\bar{x}+z-x_0|^2 - |\bar{x}-x_0|^2) K_2(z) \, dz \\ &\leq Nr_1^{-2}\|u\|_{L_\infty(B_R)} \int_{B_{r_1/2}} (|z|^2 + 2|z||\bar{x}-x_0|) J(z) \, dz \\ &\leq Nr_1^{-1}\|u\|_{L_\infty(B_R)} \int_{B_{r_1/2}} |z| J(z) \, dz. \end{aligned}$$

So

$$\mathcal{I}_2 \leq N\|u\|_{L_\infty(B_R)} \left(j(r_1/2)r_1^d + r_1^{-1} \int_{B_{r_1/2}} |z| J(z) \, dz \right). \quad (4.61)$$

By combining (4.50), (4.54), (4.60) and (4.61),

$$\begin{aligned} 0 &\leq N_1 \left(\text{osc}_{B_R} f + \frac{j(r_1)}{j(R)} \|u\|_{L_1(\mathbb{R}^d, w_R)} \right. \\ &\quad \left. + \|u\|_{L_\infty(B_R)} \left[j(r_1/2)r_1^d + r_1^{-1} \int_{B_{r_1/2}} |z| J(z) \, dz \right] \right) \\ &\quad - 2^{\alpha-3}C\alpha(1-\alpha) \int_{\mathcal{C}} |b|^{\alpha-2}|z|^2 K_1(z) \, dz. \end{aligned}$$

Thus, if $C \geq C_1 := 2(r_1/2)^{-\alpha}\|u\|_{L_\infty(B_R)}$ and

$$\begin{aligned} C &\geq C_2 := N_1 C_3 \left(\text{osc}_{B_R} f + \frac{j(r_1)}{j(R)} \|u\|_{L_1(\mathbb{R}^d, w_R)} \right. \\ &\quad \left. + \|u\|_{L_\infty(B_R)} \left[j(r_1/2)r_1^d + r_1^{-1} \int_{B_{r_1/2}} |z| J(z) \, dz \right] \right), \end{aligned}$$

then

$$\begin{aligned}
0 &\leq N_1 \left(\text{osc}_{B_R} f + \frac{j(r_1)}{j(R)} \|u\|_{L_1(\mathbb{R}^d, w_R)} \right. \\
&\quad \left. + \|u\|_{L_\infty(B_R)} \left[j(r_1/2) r_1^d + r_1^{-1} \int_{B_{r_1/2}} |z| J(z) dz \right] \right) \\
&\quad \times \left(1 - C_3 2^{\alpha-3} \alpha (1 - \alpha) \int_{\mathcal{C}} |b|^{\alpha-2} |z|^2 K_1(z) dz \right) \\
&:= (1 - C_3 C_4(b)).
\end{aligned}$$

If we take C_3 so that $C_3 = 1/C_5$ for a $C_5 = C_5(r_1, \alpha) < C_4(b)$ which does not depend on b and will be chosen below, we get the contradiction. To select C_5 , observe that with **H1** and the fact $|b| \leq r_1/2$

$$\begin{aligned}
C_4(b) &= 2^{\alpha-3} \alpha (1 - \alpha) \int_{\mathcal{C}} |b|^{\alpha-2} |z|^2 K_1(z) dz \\
&\geq \nu 2^{\alpha-3} \alpha (1 - \alpha) \int_{\mathcal{C}} |b|^{\alpha-2} |z|^2 J(z) dz \\
&\geq \kappa_1^{-1} \nu 2^{\alpha-3} \alpha (1 - \alpha) j(\eta_1 |b|) \int_{\mathcal{C}} |b|^{\alpha-2} |z|^2 dz \\
&\geq \kappa_1^{-1} \nu 2^{\alpha-3} \alpha (1 - \alpha) j(\eta_1 |b|) |b|^{\alpha-2} |\eta_1 b|^{d+2} \int_{\mathcal{C}_{\eta_2}} |z|^2 dz \\
&\geq \kappa_1^{-1} \nu \eta_1^{d+2} 2^{\alpha-3} \alpha (1 - \alpha) j(|b|) |b|^{d+\alpha} \int_{\mathcal{C}_{\eta_2}} |z|^2 dz \\
&\geq \kappa_1^{-2} \nu j(r_1/2) (r_1/2)^{d+\alpha} \eta_1^{d+2} 2^{\alpha-3} \alpha (1 - \alpha) \int_{\mathcal{C}_{\eta_2}} |z|^2 dz \\
&= j(r_1/2) r_1^{d+\alpha} N(\alpha, \eta_1, \eta_2) := C_5,
\end{aligned}$$

where $\mathcal{C} = \{|z| < \eta_1 |b| : |z \cdot b| \geq (1 - \eta_2) |b| |z|\}$ and $\mathcal{C}_{\eta_2} = \{|z| < 1 : \frac{|z \cdot b|}{|b| |z|} \geq (1 - \eta_2)\}$. Therefore, (4.51) holds with $C = C_1 + C_2$. Since C is independent of x_0 , (4.47) is proved.

Next we consider **the case** $\sigma = 1$. Note that, because K_1 is symmetric, both K_1 and K_2 satisfy (2.26). Therefore, we can replace 1_{B_1} with $I_{B_{r_1}}$ in the definition of L_2 , and get $\mathcal{I}_2 = \mathcal{I}_{21} + \mathcal{I}_{22}$, where

$$\mathcal{I}_{21} = \int_{|z| \geq r_1/2} (w(\bar{x} + z) - w(\bar{x}) - w(\bar{y} + z) + w(\bar{y})) K_2(z) dz,$$

$$\mathcal{I}_{22} = \int_{B_{r_1/2}} (w(\bar{x} + z) - w(\bar{x}) - w(\bar{y} + z) + w(\bar{y}) - z \cdot (\nabla w(\bar{x}) - \nabla w(\bar{y}))) K_2(z) dz.$$

\mathcal{I}_{21} is already estimated in the previous case. Thus we only consider \mathcal{I}_{22} . Since $M(x, y)$ attains its maximum at the interior point (\bar{x}, \bar{y}) , we have $\nabla_x M(\cdot, \bar{y})(\bar{x}) = 0$, $\nabla_y M(\bar{x}, \cdot)(\bar{y}) = 0$, and therefore

$$\nabla w(\bar{x}) - \nabla w(\bar{y}) = 16r_1^{-2} \|u\|_{L_\infty(B_R)} (\bar{x} - x_0). \quad (4.62)$$

We use (4.55) and (4.62) to get

$$\begin{aligned}\mathcal{I}_{22} &\leq 8r_1^{-2}\|u\|_{L_\infty(B_R)} \int_{B_{r_1/2}} |z|^2 K_2(z) \, dz \\ &\leq 8r_1^{-2} \int_{B_{r_1/2}} |z|^2 J(z) \, dz \|u\|_{L_\infty(B_R)}.\end{aligned}$$

Therefore, (4.47) is proved following the argument in the case $\sigma < 1$.

Finally, let $\sigma > 1$. Now we have $\mathcal{I}_2 = \mathcal{I}_{21} + \mathcal{I}_{22}$, where

$$\begin{aligned}\mathcal{I}_{21} &= \int_{|z| \geq r_1/2} [w(\bar{x} + z) - w(\bar{x}) - w(\bar{y} + z) + w(\bar{y}) - z \cdot (\nabla w(\bar{x}) - \nabla w(\bar{y}))] K_2(z) \, dz, \\ \mathcal{I}_{22} &= \int_{B_{r_1/2}} [w(\bar{x} + z) - w(\bar{x}) - w(\bar{y} + z) + w(\bar{y}) - z \cdot (\nabla w(\bar{x}) - \nabla w(\bar{y}))] K_2(z) \, dz.\end{aligned}$$

Since $\sigma > 1$, $|\bar{x} - x_0| < r_1/2$, by (4.62) and **H1**

$$\begin{aligned}\mathcal{I}_{21} &\leq \int_{|z| \geq r_1/2} [4\|u\|_{L_\infty(B_R)} + 4(r_1/2)^{-1}\|u\|_{L_\infty(B_R)}|z|] K_2(z) \, dz \\ &\leq Nr_1^d j(r_1/2) \|u\|_{L_\infty(B_R)}.\end{aligned}$$

For \mathcal{I}_{22} , we apply (4.55) and (4.62) to get

$$\mathcal{I}_{22} \leq Nr_1^{-2} \|u\|_{L_\infty(B_R)} \int_{B_{r_1/2}} |z|^2 J(z) \, dz.$$

So we again argue as in the first case to get the contradiction. Hence (i) is proved.

The proof of (ii) is quite similar to that of (i). Denote the counter parts of w and g by \tilde{w} and \tilde{g} , respectively. Also we introduce \mathcal{I}_1 and \mathcal{I}_2 similarly. That is \mathcal{I}_1 is same as before, and \mathcal{I}_2 is given by

$$\begin{aligned}\mathcal{I}_2 &= \int_{|z| \geq r_1/2} \left[\tilde{w}(\bar{x} + z) - \tilde{w}(\bar{x}) - \tilde{w}(\bar{y} + z) + \tilde{w}(\bar{y}) \right. \\ &\quad \left. - I_{B_1}(z) z \cdot \nabla(\tilde{w}(\bar{x}) - \tilde{w}(\bar{y})) \right] K_2(z) \, dz \\ &\quad + \int_{B_{r_1/2}} \left[\tilde{w}(\bar{x} + z) - \tilde{w}(\bar{x}) - \tilde{w}(\bar{y} + z) + \tilde{w}(\bar{y}) \right. \\ &\quad \left. - I_{B_1}(z) z \cdot \nabla(\tilde{w}(\bar{x}) - \tilde{w}(\bar{y})) \right] K_2(z) \, dz \\ &:= \mathcal{I}_{21} + \mathcal{I}_{22}.\end{aligned}$$

All of the differences are as follows. If $r_1/2 \geq 1$, then by using (4.55) and (4.62),

$$\begin{aligned}\mathcal{I}_{22} &\leq Nr_1^{-2} \|\tilde{u}\|_{L_\infty(B_R)} \left[\int_{B_1} |z|^2 K_2(z) \, dz + \int_{1 \leq |z| \leq r_1/2} (|z|^2 + (\bar{x} - x_0) \cdot z) K_2(z) \, dz \right] \\ &\leq NI_{\sigma < 1} r_1^{-1} \|\tilde{u}\|_{L_\infty(B_R)} \int_{B_{r_1/2}} |z| J(z) \, dz \\ &\quad + NI_{\sigma > 1} r_1^{-2} \|\tilde{u}\|_{L_\infty(B_R)} \int_{B_{r_1/2}} |z|^2 J(z) \, dz.\end{aligned}$$

In the above, we also used $\int_{1 \leq |z| \leq r_1/2} z^i K_2(z) dz = 0$ if $\sigma > 1$ (due to **H3(iv)**).

Let $\sigma < 1$ and $r_1/2 < 1$. If **H3**(ii) hold, then by (2.17),

$$\begin{aligned} \mathcal{I}_{21} &\leq N \|u\|_{L_\infty(B_R)} \int_{|z| \geq r_1/2} J(z) dz \\ &= N r_1^d \int_{|z| \geq 1} J(r_1 z/2) dz \leq N j(r_1/2) r_1^d \|u\|_{L_\infty(B_R)}. \end{aligned}$$

Also, if **H3**(iii) holds, then by using (4.62),

$$\begin{aligned} \mathcal{I}_{21} &\leq \|u\|_{L_\infty(B_R)} \int_{|z| \geq r_1/2} [1 + 8r_1^{-1}|z|] K_2(z) dz \\ &\leq N j(r_1/2) r_1^d \|u\|_{L_\infty(B_R)}. \end{aligned}$$

This completes the proof of the theorem. \square

We remove $\sup_{B_R} u$ on the right hand side of (4.48) in the following corollary. Recall $w_R(x) = \frac{1}{1/j(R)+1/J(x/2)}$.

Corollary 4.3. *Suppose that **H1** and **H2** hold. Let $\lambda \geq 0$, $f \in L_\infty(B_1)$, and $u, \tilde{u} \in C_b^2(B_R) \cap L_1(\mathbb{R}^d, w_R)$ satisfy*

$$Lu - \lambda u = f, \quad \tilde{L}\tilde{u} - \lambda \tilde{u} = f \quad \text{in } B_R. \quad (4.63)$$

(i) *For any $\alpha \in (0, \min\{1, \alpha_0\})$, it holds that*

$$[u]_{C^\alpha(B_{R/2})} \leq \frac{N}{j(R)R^{d+\alpha}} (\|u\|_{L_1(\mathbb{R}^d, w_R)} + \text{osc}_{B_R} f), \quad (4.64)$$

where $N = N(d, \nu, \Lambda, \kappa_1, \alpha_0, \alpha)$.

(ii) *If one of **H3** (ii)-(iv) is additionally assumed, then (4.64) holds for \tilde{u} .*

Proof. For $n = 1, 2, \dots$, set

$$r_n := R(1 - 2^{-n}).$$

Observe that $(r_{n+1} - r_n)/2 = R2^{-n-2} \leq R$ and by **H1**

$$\begin{aligned} \frac{1}{j(r_{n+1})} \|u\|_{L_1(\mathbb{R}^d, w_{r_{n+1}})} &\leq \left(\int_{|z| < 2R} u(z) dz + \frac{1}{j(r_{n+1})} \int_{|z| \geq 2R} u(z) j(z/2) dz \right) \\ &\leq N \left(\int_{|z| < 2R} u(z) dz + \frac{1}{j(R)} \int_{|z| \geq 2R} u(z) j(z/2) dz \right) \\ &\leq N \frac{1}{j(R)} \int_{\mathbb{R}^d} u(z) w_R(z) dz. \end{aligned}$$

Then by Theorem 4.2 (i) and **H1**,

$$\begin{aligned} [u]_{C^\alpha(B_{r_n})} &\leq N R^{-\alpha} 2^{\alpha n} \sup_{B_{r_{n+1}}} |u| \\ &\quad + N \frac{2^{(d+\alpha)n}}{j(R2^{-n-2})R^{d+\alpha}} \left(\frac{j(R2^{-n-2})}{j(r_{n+1})} \|u\|_{L_1(\mathbb{R}^d, w_{r_{n+1}})} + \text{osc}_{B_{r_{n+1}}} f \right) \\ &\leq N \left[R^{-\alpha} 2^{\alpha n} \sup_{B_{r_{n+1}}} |u| + \frac{2^{(d+\alpha)n}}{j(R)R^{d+\alpha}} (\|u\|_{L_1(\mathbb{R}^d, w_R)} + \text{osc}_{B_R} f) \right]. \end{aligned} \quad (4.65)$$

In order to estimate the term $\sup_{B_{r_{n+1}}} |u|$ above, we use the following :

$$\sup_{B_{r_{n+1}}} |u| \leq (\varepsilon r_{n+1})^\alpha [u]_{C^\alpha(r_{n+1})} + N(\varepsilon r_{n+1})^{-d} \|u\|_{L_1(B_{r_{n+1}})}, \quad \varepsilon \in (0, 1). \quad (4.66)$$

Actually this inequality can be easily obtained as follows. For all $\varepsilon \in (0, 1)$, $x \in B_{r_{n+1}}$ and $y \in B_{r_{n+1}} \cap B_{\varepsilon r_{n+1}}(x)$,

$$\begin{aligned} & |B_{r_{n+1}} \cap B_{\varepsilon r_{n+1}}(x)| \cdot |u(x)| \\ & \leq \int_{B_{r_{n+1}} \cap B_{\varepsilon r_{n+1}}(x)} (|u(x) - u(y)| + |u(y)|) \, dy \\ & \leq |B_{r_{n+1}} \cap B_{\varepsilon r_{n+1}}(x)| \cdot (\varepsilon r_{n+1})^\alpha [u]_{C^\alpha(B_{r_{n+1}})} + \int_{B_{r_{n+1}} \cap B_{\varepsilon r_{n+1}}(x)} |u(y)| \, dy. \end{aligned}$$

Now it is enough to note that $|B_{r_{n+1}} \cap B_{\varepsilon r_{n+1}}(x)| \sim (\varepsilon r_{n+1})^d$ because $\varepsilon \in (0, 1)$ and $x \in B_{r_{n+1}}$.

Take N from (4.65) and define ε so that

$$\varepsilon^\alpha = N^{-1} 2^{-\alpha n} 2^{-3d}.$$

Then by combining (4.65) and (4.66),

$$\begin{aligned} [u]_{C^\alpha(B_{r_n})} & \leq 2^{-3d} [u]_{C^\alpha(B_{r_{n+1}})} + N R^{-d-\alpha} 2^{2dn} \|u\|_{L_1(B_{r_{n+1}})} \\ & \quad + N \frac{2^{(d+\alpha)n}}{j(R) R^{d+\alpha}} (\|u\|_{L_1(\mathbb{R}^d, w_R)} + \text{osc}_{B_R} f) \\ & \leq 2^{-3d} [u]_{C^\alpha(B_{r_{n+1}})} + N R^{-d-\alpha} 2^{2dn} \|u\|_{L_1(B_{r_{n+1}})} \\ & \quad + N \frac{2^{2dn}}{j(R) R^{d+\alpha}} (\|u\|_{L_1(\mathbb{R}^d, w_R)} + \text{osc}_{B_R} f). \end{aligned} \quad (4.67)$$

Multiply both sides of (4.67) by 2^{-3dn} and take the sum over n to get

$$\begin{aligned} & \sum_{n=1}^{\infty} 2^{-3dn} [u]_{C^\alpha(B_{r_n})} \\ & \leq \sum_{n=1}^{\infty} 2^{-3d(n+1)} [u]_{C^\alpha(B_{r_{n+1}})} + N \sum_{n=1}^{\infty} 2^{-dn} R^{-d-\alpha} \|u\|_{L_1(B_{r_{n+1}})} \\ & \quad + N \left(\sum_{n=1}^{\infty} 2^{-dn} \right) \frac{1}{j(R) R^{d+\alpha}} (\|u\|_{L_1(\mathbb{R}^d, w_R)} + N \text{osc}_{B_R} f). \end{aligned}$$

Since $[u]_{C^\alpha(B_{r_n})} \leq [u]_{C^\alpha(B_R)} < \infty$ and by **H1**

$$\|u\|_{L_1(B_{r_{n+1}})} \leq \|u\|_{L_1(B_R)} = \frac{j(R)}{j(R)} \|u\|_{L_1(B_R)} \leq \frac{N}{j(R)} \|u\|_{L_1(\mathbb{R}^d, w_R)},$$

(i) is proved.

(ii) is proved similarly by following the proof of (i) with Theorem 4.2 (ii). \square

5. SOME SHARP FUNCTION AND MAXIMAL FUNCTION ESTIMATES

For $g \in L_{1,\text{loc}}(\mathbb{R}^d)$, the maximal function and sharp function are defined as follows :

$$\mathcal{M}g(x) := \sup_{r>0} \int_{B_r(x)} |g(y)| \, dy := \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |g(y)| \, dy,$$

and

$$g^\#(x) := \sup_{r>0} \int_{B_r(x)} |g(y) - (g)_{B_r(x)}| \, dy := \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |g(y) - (g)_{B_r(x)}| \, dy,$$

where $(g)_{B_r(x)} = \frac{1}{|B_r(x)|} \int_{B_r(x)} g(y) \, dy$ the average of g on $B_r(x)$.

Lemma 5.1. *Suppose that **H1** and **H2** hold. Let $\lambda \geq 0$, $R > 0$, $f \in C_0^\infty$, and $f = 0$ in B_{2R} . Assume that $u, \tilde{u} \in H_2^A \cap C_b^\infty$ satisfy*

$$Lu - \lambda u = f, \quad \tilde{L}\tilde{u} - \lambda \tilde{u} = f. \quad (5.68)$$

(i) *Then for all $\alpha \in (0, \min\{1, \alpha_0\})$,*

$$[u]_{C^\alpha(B_{R/2})} \leq NR^{-\alpha} \sum_{k=1}^{\infty} 2^{-\alpha_0 k} (|u|)_{B_{2^k R}}, \quad (5.69)$$

$$[\mathcal{A}u]_{C^\alpha(B_{R/2})} \leq NR^{-\alpha} \left(\sum_{k=1}^{\infty} 2^{-\alpha_0 k} (|\mathcal{A}u|)_{B_{2^k R}} + \mathcal{M}f(0) \right), \quad (5.70)$$

where N depends only on $d, \nu, \Lambda, \kappa_1, \kappa_2, \alpha_0$, and α .

(ii) *If one of **H3**(ii)-(iv) is additionally assumed, then (5.69) and (5.70) hold for \tilde{u} .*

Proof. By Corollary 4.3 and the assumption that $f = 0$ in B_{2R} ,

$$[u]_{C^\alpha(B_{R/2})} \leq N \frac{1}{j(R)R^{d+\alpha}} \|u\|_{L_1(\mathbb{R}^d, w_R)}. \quad (5.71)$$

Set

$$B_{(0)} = B_R, \quad B_{(k)} = B_{2^k R} \setminus B_{2^{k-1} R}, \quad k \geq 1.$$

Observe that

$$\begin{aligned} \|u\|_{L_1(\mathbb{R}^d, w_R)} &= \int_{\mathbb{R}^d} |u(y)| \frac{1}{1/j(R) + 1/j(|y|/2)} \, dy \\ &= \sum_{k=0}^{\infty} \int_{B_{(k)}} |u(y)| \frac{1}{1/j(R) + 1/j(|y|/2)} \, dy \\ &\leq 2j(R) \int_{B_{2R}} |u(y)| \, dy + N \sum_{k=2}^{\infty} j(2^{k-2}R) \int_{B_{2^k R}} |u(y)| \, dy \\ &\leq N \left(j(R)R^d (|u|)_{B_{2R}} + \sum_{k=2}^{\infty} 2^{-(k-2)(d+\alpha_0)} j(R) \int_{B_{2^k R}} |u(y)| \, dy \right) \\ &\leq N \left(j(R)R^d (|u|)_{B_{2R}} + \sum_{k=2}^{\infty} 2^{-(k-2)(d+\alpha_0)} 2^{kd} j(R)R^d (|u|)_{B_{2^k R}} \right) \\ &\leq Nj(R)R^d \left(\sum_{k=1}^{\infty} 2^{-\alpha_0 k} (|u|)_{B_{2^k R}} \right), \end{aligned}$$

where the first and second inequalities come from **H1**. Therefore we get (5.69).

To prove (5.70), we apply the operator \mathcal{A} to both sides of $Lu - \lambda u = f$ and obtain

$$(L - \lambda)(\mathcal{A}u) = \mathcal{A}f.$$

By applying Corollary 4.3 again,

$$[\mathcal{A}u]_{C^\alpha(B_{R/2})} \leq N \frac{1}{j(R)R^{d+\alpha}} \left(\|\mathcal{A}u\|_{L_1(\mathbb{R}^d, w_R)} + \sup_{B_R} |\mathcal{A}f| \right). \quad (5.72)$$

The first term on the right hand side of (5.72) is bounded by

$$NR^{-\alpha} \left(\sum_{k=0}^{\infty} 2^{-\alpha_0 k} (|\mathcal{A}u|)_{B_{2^k R}} \right).$$

In order to estimate the second term, we recall the definition of \mathcal{A} . For $|x| < R$,

$$\begin{aligned} |\mathcal{A}f(x)| &= \left| \int_{\mathbb{R}^d} [f(x+y) - f(x)] J(y) dy \right| \\ &\leq \sum_{k=1}^{\infty} \int_{B_{(k)}} |f(x+y)| j(|y|) dy \\ &\leq N \sum_{k=1}^{\infty} j(2^{k-1}R) \int_{B_{(k)}} |f(x+y)| dy \\ &\leq N \sum_{k=1}^{\infty} 2^{-(k-1)(d+\alpha_0)} j(R) \int_{B_{2^k R}} |f(x+y)| dy \\ &\leq N \sum_{k=1}^{\infty} 2^{-(k-1)(d+\alpha_0)} j(R) \int_{B_{2^{k+1}R}} |f(y)| dy \\ &\leq N j(R) R^d \left(\sum_{k=1}^{\infty} 2^{-\alpha_0 k} (|f|)_{B_{2^{k+1}R}} \right) \leq N j(R) R^d \mathcal{M}f(0), \end{aligned}$$

where the first inequality is due to the assumption $f(x) = 0$ if $|x| < 2R$ and both the second and the third inequality are owing to **H1**. Therefore (i) is proved. Also, (ii) is proved similarly with Corollary 4.3 (ii). \square

The above lemma easily yields the following mean oscillation estimate.

Corollary 5.2. *Suppose that **H1** and **H2** hold. Let $\lambda \geq 0$ and $r, \kappa > 0$. Assume $f \in C_0^\infty$, $f = 0$ in $B_{2\kappa r}$, and $u, \tilde{u} \in H_2^A \cap C_b^\infty$ satisfy*

$$Lu - \lambda u = f, \quad \tilde{L}\tilde{u} - \lambda \tilde{u} = f.$$

(i) *Then for all $\alpha \in (0, \min\{1, \alpha_0\})$,*

$$(|u - (u)_{B_r}|)_{B_r} \leq N \kappa^{-\alpha} \sum_{k=1}^{\infty} 2^{-\alpha_0 k} |u|_{B_{2^k \kappa r}}, \quad (5.73)$$

$$(|\mathcal{A}u - \mathcal{A}u|_{B_r})_{B_r} \leq N \kappa^{-\alpha} \left(\sum_{k=1}^{\infty} 2^{-\alpha_0 k} |\mathcal{A}u|_{B_{2^k \kappa r}} + \mathcal{M}f(0) \right), \quad (5.74)$$

where N depends only on $d, \nu, \Lambda, \kappa_1, \kappa_2, \alpha_0$, and α .

(ii) If one of **H3** (ii)-(iv) is additionally assumed, then (5.73) and (5.74) hold for \tilde{u} .

Proof. It is enough to use the following inequality

$$(|u - (u)_{B_r}|)_{B_r} \leq 2^\alpha r^\alpha [u]_{C^\alpha(r)} \leq 2^\alpha r^\alpha [u]_{C^\alpha(\kappa r/2)}$$

and apply Lemma 5.1 with $R = \kappa r$. \square

Next we show that the mean oscillation of u is controlled by the maximal functions of u and $Lu - \lambda u$.

Lemma 5.3. *Suppose that **H1** and **H2** hold. Let $\lambda > 0$, $\kappa \geq 2$, $r > 0$, and $f \in C_0^\infty$. Assume $u, \tilde{u} \in H_2^A \cap C_b^\infty$ satisfy*

$$Lu - \lambda u = f, \quad \tilde{L}u - \lambda u = f. \quad (5.75)$$

(i) Then for all $\alpha \in (0, \min\{1, \alpha_0\})$,

$$\begin{aligned} & \lambda(|u - (u)_{B_r}|)_{B_r} + (|\mathcal{A}u - (\mathcal{A}u)_{B_r}|)_{B_r} \\ & \leq N\kappa^{-\alpha} (\lambda \mathcal{M}u(0) + \mathcal{M}(\mathcal{A}u)(0)) + N\kappa^{d/2} (\mathcal{M}(f^2)(0))^{1/2}, \end{aligned} \quad (5.76)$$

where N depends only on d, ν, Λ , and J .

(ii) If one of **H3** (ii)-(iv) is additionally assumed, then (5.76) holds for \tilde{u} .

Proof. Due to the similarity of the proof, we only prove the assertion (i).

Take a cut-off function $\eta \in C_0^\infty(B_{4\kappa r})$ satisfying $\eta = 1$ in $B_{2\kappa r}$. By Theorem 2.8, there exists a unique solution u in H_2^A satisfying

$$Lw - \lambda w = \eta f \quad (5.77)$$

and

$$\lambda\|w\|_{L_2} + \|\mathcal{A}w\|_{L_2} \leq N\|\eta f\|_{L_2}. \quad (5.78)$$

From (5.78), Jensen's inequality, and the fact ηf has its support within $B_{4\kappa r}$, for any $R > 0$,

$$\begin{aligned} \lambda(|w|)_{B_R} + (|\mathcal{A}w|)_{B_R} & \leq NR^{-d/2} (\lambda\|w\|_{L_2} + \|\mathcal{A}w\|_{L_2}) \\ & \leq NR^{-d/2} \|\eta f\|_{L_2} \\ & \leq NR^{-d/2} (\kappa r)^{d/2} (\mathcal{M}(f^2)(0))^{1/2}. \end{aligned} \quad (5.79)$$

Furthermore, taking $(1-\Delta)^\gamma$ to both sides of (5.77) and using the fact $(1-\Delta)^\gamma Lw = L(1-\Delta)^\gamma w$, we can easily check that $w \in C_b^\infty$ by Sobolev's inequality. By setting $v := u - w$, from (5.77) and (5.75)

$$Lv - \lambda v = (1-\eta)f, \quad v \in C_b^\infty \cap H_2^A.$$

By applying Corollary 5.2 to v ,

$$\begin{aligned}
& (\lambda|v - (v)_{B_r}|)_{B_r} + (|\mathcal{A}v - (\mathcal{A}v)_{B_r}|)_{B_r} \\
& \leq N\kappa^{-\alpha} \left(\sum_{k=1}^{\infty} 2^{-\alpha_0 k} [\lambda(|v|)_{B_{2^k \kappa r}} + (|\mathcal{A}v|)_{B_{2^k \kappa r}}] + \mathcal{M}f(0) \right) \\
& \leq N\kappa^{-\alpha} \left(\sum_{k=1}^{\infty} 2^{-\alpha_0 k} [\lambda(|u|)_{B_{2^k \kappa r}} + (|\mathcal{A}u|)_{B_{2^k \kappa r}}] \right) \\
& \quad + N\kappa^{-\alpha} \left(\sum_{k=0}^{\infty} 2^{-\alpha_0 k} [\lambda(|w|)_{B_{2^k \kappa r}} + (|\mathcal{A}w|)_{B_{2^k \kappa r}}] + \mathcal{M}f(0) \right) \\
& \leq N\kappa^{-\alpha} \left(\sum_{k=1}^{\infty} 2^{-\alpha_0 k} [\lambda(|u|)_{B_{2^k \kappa r}} + (|\mathcal{A}u|)_{B_{2^k \kappa r}}] \right) \\
& \quad + N\kappa^{-\alpha} \left(\sum_{k=1}^{\infty} 2^{-\alpha_0 k} [2^{-dk/2} (\mathcal{M}(f^2)(0))^{1/2}] + \mathcal{M}f(0) \right) \\
& \leq N\kappa^{-\alpha} \left(\lambda \mathcal{M}u(0) + \mathcal{M}(\mathcal{A}u)(0) + (\mathcal{M}(f^2)(0))^{1/2} \right), \tag{5.80}
\end{aligned}$$

where (5.79) is used for the third inequality with $R = 2^k \kappa r$, and for the last inequality we use $\mathcal{M}f(0) \leq (\mathcal{M}(f^2)(0))^{1/2}$. By combining (5.79) and (5.80),

$$\begin{aligned}
& \lambda(|u - (u)_{B_r}|)_{B_r} + (|\mathcal{A}u - (\mathcal{A}u)_{B_r}|)_{B_r} \\
& \leq N(\lambda|v - (v)_{B_r}|)_{B_r} + (|\mathcal{A}v - (\mathcal{A}v)_{B_r}|)_{B_r} + \lambda(|w|)_{B_r} + (|\mathcal{A}w|)_{B_r} \\
& \leq N\kappa^{-\alpha} (\lambda \mathcal{M}u(0)) + N\mathcal{M}(\mathcal{A}u)(0) + N(\mathcal{M}(f^2)(0))^{1/2}.
\end{aligned}$$

Therefore, the lemma is proved. \square

We make full use of Lemma 5.1 to get the mean oscillation of Lu .

Lemma 5.4. *Suppose that **H1** and **H2** hold. Let $\lambda > 0$, $\kappa \geq 2$, $r > 0$, and $f \in C_0^\infty$. Assume $u \in H_2^A \cap C_b^\infty$ satisfy*

$$\mathcal{A}u - \lambda u = f.$$

Then for all $\alpha \in (0, \min\{1, \alpha_0\})$,

$$\begin{aligned}
& \lambda(|u - (u)_{B_r}|)_{B_r} + (|Lu - (Lu)_{B_r}|)_{B_r} \\
& \leq N\kappa^{-\alpha} (\lambda \mathcal{M}u(0) + \mathcal{M}(Lu)(0)) + N\kappa^{d/2} (\mathcal{M}(f^2)(0))^{1/2},
\end{aligned}$$

where N depends only on d, ν, Λ , and J .

Proof. Exchanging the roles of \mathcal{A} and L in the proof of Lemma 5.1, we easily get

$$[Lu]_{C^\alpha(B_{R/2})} \leq NR^{-\alpha} \left(\sum_{k=0}^{\infty} 2^{-\alpha_0 k} (Lu)_{B_{2^k R}} + \mathcal{M}f(0) \right).$$

Therefore, the lemma is proved as we follow the proof of Lemma 5.3. \square

6. PROOF OF THEOREMS 2.16 AND 2.21

Proof of Theorem 2.16

The case $p = 2$ was already proved in Theorem 2.8. Due to Corollary 2.7 and Lemmas 2.6, it is sufficient to prove

$$\|\mathcal{A}u\|_{L_p} + \lambda\|u\|_{L_p} \leq N\|Lu - \lambda u\|_{L_p}, \quad \forall u \in C_0^\infty, \quad (6.81)$$

where $N = N(d, \nu, \Lambda, \kappa_1, \kappa_2, \alpha_0)$.

First, assume $p > 2$. Put $f := Lu - \lambda u$. From Lemma 5.3, for all $\alpha \in (0, \min\{1, \alpha_0\})$

$$\begin{aligned} & \lambda(|u - (u)_{B_r}|)_{B_r} + (|\mathcal{A}u - (\mathcal{A}u)_{B_r}|)_{B_r} \\ & \leq N\kappa^{-\alpha}(\lambda\mathcal{M}u(0) + \mathcal{M}(\mathcal{A}u)(0)) + N\kappa^{d/2}(\mathcal{M}(f^2)(0))^{1/2}. \end{aligned}$$

By translation, it is easy to check that the above inequality holds for all $B_r(x)$ with $x \in \mathbb{R}^d$ and $r > 0$. By the arbitrariness of r ,

$$\begin{aligned} & \lambda u^\#(x) + (\mathcal{A}u)^\#(x) \\ & \leq N\kappa^{-\alpha}(\lambda\mathcal{M}u(x) + \mathcal{M}(\mathcal{A}u)(x)) + N\kappa^{d/2}(\mathcal{M}(f^2)(x))^{1/2}. \end{aligned}$$

Therefore, by the Fefferman-Stein theorem and Hardy-Littlewood maximal theorem (see, for instance, chapter 1 of [15]), we get

$$\lambda\|u\|_{L_p} + \|\mathcal{A}u\|_{L_p} \leq N\kappa^{-\alpha}(\lambda\|u\|_{L_p} + \|\mathcal{A}u\|_{L_p}) + N\kappa^{d/2}\|f\|_{L_p}.$$

By choosing $\kappa > 2$ large enough so that $N\kappa^{-\alpha} < 1/2$,

$$\lambda\|u\|_{L_p} + \|\mathcal{A}u\|_{L_p} \leq N\|f\|_{L_p}.$$

We use the duality argument for $p \in (1, 2)$. Put $q := p/(p-1)$. Then since $q \in (2, \infty)$, for any $g \in C_0^\infty$ there is a unique $v_g \in H_q^{\mathcal{A}}$ satisfying

$$L^*v_g - \lambda v_g = g \quad \text{in } \mathbb{R}^d.$$

Therefore, by applying (6.81) with $q \in (2, \infty)$, for any $u \in C_0^\infty$,

$$\begin{aligned} \|\mathcal{A}u\|_{L_p} & \leq \sup_{\|g\|_{L_q}=1, g \in C_0^\infty} \int_{\mathbb{R}^d} |g\mathcal{A}u| \, dx \\ & = \sup_{\|g\|_{L_q}=1, g \in C_0^\infty} \int_{\mathbb{R}^d} |(L^*v_g - \lambda v_g)\mathcal{A}u| \, dx \\ & = \sup_{\|g\|_{L_q}=1, g \in C_0^\infty} \int_{\mathbb{R}^d} |\mathcal{A}v_g(Lu - \lambda u)| \, dx \\ & \leq \sup_{\|g\|_{L_q}=1, g \in C_0^\infty} \|\mathcal{A}v_g\|_{L_q} \|Lu - \lambda u\|_{L_p} \\ & \leq \sup_{\|g\|_{L_q}=1, g \in C_0^\infty} N\|g\|_{L_q} \|Lu - \lambda u\|_{L_p} = N\|Lu - \lambda u\|_{L_p}. \end{aligned}$$

Similarly,

$$\lambda\|u\|_{L_p} \leq N\|Lu - \lambda u\|_{L_p}.$$

Finally, we prove the continuity of the operator L by showing

$$\|Lu\|_{L_p} \leq N\|\mathcal{A}u\|_{L_p}, \quad \forall u \in C_0^\infty. \quad (6.82)$$

Recall that we proved (6.81) based on Lemma 5.3. Similarly, using Lemma 5.4, one can prove

$$\|Lu\|_{L_p} \leq N\|\mathcal{A}u - \lambda u\|_{L_p} \quad \forall u \in C_0^\infty, \quad \forall \lambda > 0.$$

Since N is independent of λ , this leads to (6.82). The theorem is proved.

Proof of Theorem 2.21

The proof is identical to that of Theorem 2.16 if one of **H3(ii)**-(iv) holds. So it only remains to prove

$$\|\mathcal{A}u\|_{L_p} + \lambda\|u\|_{L_p} \leq N\|\tilde{L}u - \lambda u\|_{L_p}, \quad \forall u \in C_0^\infty$$

under the condition **H3(i)**. Define

$$b^i = - \int_{B_1} y^i a(y) J(y) dy \quad \text{if } \sigma \in (0, 1), \quad b^i = \int_{\mathbb{R}^d \setminus B_1} y^i a(y) J(y) dy \quad \text{if } \sigma \in (1, 2).$$

Then under **H1** and **H2**, $|b| < \infty$ and for any $u \in C_0^\infty$, we have

$$\tilde{L}u = Lu + b \cdot \nabla u,$$

and therefore

$$\|u\|_{L_p} + \|\mathcal{A}u\|_{L_p} \leq N\|Lu - \lambda u\|_{L_p} \leq N(\|\tilde{L}u - \lambda u\|_{L_p} + \|\nabla u\|_{L_p}).$$

Take $\varepsilon = 1/(2N)$ in **H3(i)** and apply Lemma 2.1. Then, the theorem is proved. \square

REFERENCES

- [1] D. Applebaum, *Lévy processes and stochastic calculus*, Cambridge University Press, 2009.
- [2] R. Bañuelos and K. Bogdan, *Lévy processes and Fourier multipliers*, J. Funct. Anal. **250** (2007), no. 1, 197–213.
- [3] G. Barles, E. Chasseigne, and C. Imbert, *Hölder continuity of solutions of second-order non-linear elliptic integro-differential equations. (English summary)* J. Eur. Math. Soc. (JEMS) **13** (2011), no. 1, 1–26.
- [4] R.F. Bass, *Harnack inequalities for non-local operators of variable order*, Trans. Amer. Math. Soc. **357** (2005), no. 2, 837–850.
- [5] R.F. Bass, *Hölder continuity of harmonic functions with respect to operators of variable order*, Comm. Partial Differential Equations **30** (2005), 1249–1259.
- [6] J. Bertoin, *Lévy processes*, Cambridge University Press, Cambridge, 1996.
- [7] Z. Q. Chen, P. Kim, and R. Song, *Sharp heat kernel estimates for relativistic stable processes in open sets*, Ann. Probab. **40** (2012), no. 1, 213–244.
- [8] H. Dong and D. Kim, *Schauder estimates for a class of non-local elliptic equations*, Discrete Contin. Dyn. Syst. **33** (2013), no. 6, 2319–2347.
- [9] H. Dong and D. Kim, *On L_p -estimates for a class of non-local elliptic equations*, J. Funct. Anal. **262** (2012), no. 3, 1166–1199.
- [10] P. Kim, R. Song, and Z. Vondraček, *Global uniform boundary Harnack principle with explicit decay rate and its application*, arXiv:1212.3092 [math.PR] (2012)
- [11] N.V. Krylov, *Lectures on Elliptic and Parabolic Equations in Sobolev Spaces*, American Mathematical Society, 2008.
- [12] R. Mikulevicius and H. Pragarauskas, *On the Cauchy problem for certain integro-differential operators in Sobolev and Hölder spaces*, Liet. Mat. Rink. **32** (1992), no. 2, 299–331.
- [13] R. L. Schilling, R. Song and Z. Vondraček, *Bernstein Functions: Theory and Applications*, de Gruyter Studies in Mathematics 37. Berlin: Walter de Gruyter, 2010.
- [14] L. Silvestre, *Hölder estimates for solutions of integro-differential equations like the fractional Laplace*, Indiana Univ. Math. J. **55** (2006), no. 3, 1155–1174.
- [15] E. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, 1993.
- [16] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton. N.J, 1970.
- [17] X. Zhang, *L^p -maximal regularity of nonlocal parabolic equation and applications*, arXiv:1109.0816v4.

DEPARTMENT OF MATHEMATICS, KOREA UNIVERSITY, 1 ANAM-DONG, SUNGBUK-GU, SEOUL,
136-701, REPUBLIC OF KOREA

E-mail address: `waldoo@korea.ac.kr`

DEPARTMENT OF MATHEMATICS, KOREA UNIVERSITY, 1 ANAM-DONG, SUNGBUK-GU, SEOUL,
136-701, REPUBLIC OF KOREA

E-mail address: `kyeonghun@korea.ac.kr`